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The space $BV(S^2, S^1)$: minimal connection and optimal lifting

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Abstract

We show that topological singularities of maps in $BV(S^2, S^1)$ can be detected by its distributional Jacobian. As an application, we construct an optimal lifting and we compute its total variation.

Résumé

On montre que le jacobien d'une fonction $u \in BV(S^2, S^1)$ permet de localiser les singularités topologiques de u. On applique ce résultat à la construction d'un relèvement optimal et on calcule sa variation totale.

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1. Introduction

Let $u \in BV(S^2, S^1)$, i.e. $u = (u_1, u_2) \in L^1(S^2, \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in S^2$ and the derivative of u (in the sense of the distributions) is a finite 2×2-matrix Radon measure

$$\int_{S^2} |Du| = \sup\left\{ \int_{S^2} \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, d\mathcal{H}^2 \colon \ \zeta_k \in C^1(S^2, \mathbb{R}^2), \ \sum_{k=1}^{2} \left|\zeta_k(x)\right|^2 \leqslant 1, \ \forall x \in S^2 \right\} < \infty,$$

where the norm in $\mathbb{R}^2$ is the Euclidean norm. Observe that the total variation of Du is independent of the choice of the orthonormal frame (x, y) on S^2; a frame (x, y) is always taken such that (x, y, e) is direct, where e is the outward normal to the sphere S^2.

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We begin with the notion of minimal connection between point singularities of u. The concept of a minimal connection associated to a function from $\mathbb{R}^3$ into S^2 was originally introduced by Brezis, Coron and Lieb [3]. Following the ideas in [3] and [6], Brezis, Mironescu and Ponce [4] studied the topological singularities of functions $g \in W^{1,1}(S^2, S^1)$. They show that the distributional Jacobian of g describes the location and the topological charge of the singular set of g. More precisely, let $T(g) \in \mathcal{D}'(S^2, \mathbb{R})$ be defined as

$$T(g) = 2\det(\nabla g) = -(g \wedge g_x)_y + (g \wedge g_y)_x;$$

then there exist two sequences of points (p_k), (n_k) in S^2 such that

$$\sum_k |p_k - n_k| < \infty \quad \text{and} \quad T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

Our aim is to extend these notions for functions $u \in BV(S^2, S^1)$. In this case, the difficulty of the analysis of the singular set arises from the existence of more than one type of singularity: besides the point singularities carrying a degree, the jump singularities of u should be taken into account.

We start by introducing some notation. Write the finite Radon 2×2-matrix measure Du as

$$Du = D^a u + D^c u + D^j u,$$

where $D^a u$, $D^c u$ and $D^j u$ are the absolutely continuous part, the Cantor part and the jump part of Du (see e.g. [1]). We recall that $D^j u$ can be written as

$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^1 \llcorner S(u),$$

where $S(u)$ denotes the set of jump points of u; $S(u)$ is a countably $\mathcal{H}^1$-rectifiable set on S^2 oriented by the Borel map $\nu_u : S(u) \to S^1$. The Borel functions $u^+, u^- : S(u) \to S^1$ are the traces of u on the jump set $S(u)$ with respect to the orientation ν_u. Throughout the paper we identify u by its precise representative that is defined $\mathcal{H}^1$-a.e. in $S^2 \setminus S(u)$.

We now introduce the distribution $T(u) \in \mathcal{D}'(S^2, \mathbb{R})$ as

$$\langle T(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot \big(u \wedge (D^a u + D^c u)\big) + \int_{S(u)} \rho(u^+, u^-) \nu_u \cdot \nabla^\perp \zeta \, d\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2, \mathbb{R}). \tag{1}$$

Here, $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = (u \wedge a, u \wedge b) = (u_1 a_2 - u_2 a_1, u_1 b_2 - u_2 b_1),$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The function $\rho(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ is the signed geodesic distance on S^1 defined as

$$\rho(\omega_1, \omega_2) = \begin{cases} \operatorname{Arg}(\frac{\omega_1}{\omega_2}) & \text{if } \frac{\omega_1}{\omega_2} \neq -1, \\ \operatorname{Arg}(\omega_1) - \operatorname{Arg}(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1, \end{cases} \quad \forall \omega_1, \omega_2 \in S^1,$$

where $\operatorname{Arg}(\omega) \in (-\pi, \pi]$ stands for the argument of the unit complex number $\omega \in S^1$. $T(u)$ represents the distributional determinant of the absolutely continuous part and the Cantor part of Du which is adjusted on $S(u)$ by the tangential derivative of $\rho(u^+, u^-)$. The second term in the RHS of (1) is motivated by the study of $BV(S^1, S^1)$ functions (see [9]): we defined there a similar quantity that represents a pseudo-degree for $BV(S^1, S^1)$ functions.

Remark 1. (i) The integrand in (1) is computed pointwise in any orthonormal frame (x, y) and the corresponding quantity is frame-invariant.