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Stratified semiconcave control-Lyapunov functions and the stabilization problem

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Abstract

Given a globally asymptotically controllable control system, we construct a control-Lyapunov function which is stratified semiconcave; that is, roughly speaking whose singular set has a Whitney stratification. Then we deduce the existence of smooth feedbacks which make the closed-loop system *almost* globally asymptotically stable.

Keywords: Control-Lyapunov function; Semiconcave function; Feedback stabilization

MSC: 93D05; 93D20; 93B05; 34D20; 49J52; 49L25; 70K15

Introduction

This paper is concerned with the stabilization problem for control systems of the form

$$\dot{x} = f(x,\alpha) := \sum_{i=1}^{m} \alpha_i f_i(x), \tag{1}$$

where $f_1, \dots, f_m$ are smooth vector fields on $\mathbb{R}^N$ and where the control $\alpha = (\alpha_1, \dots, \alpha_m)$ belongs to $\overline{B_m}$ the closed unit ball in $\mathbb{R}^m$. We focus on control systems which are globally asymptotically controllable.

Definition 1. The system (1) is said to be globally asymptotically controllable (abreviated GAC) if the two following conditions are satisfied:

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1. (Attractivity) For each $x \in \mathbb{R}^N$ there exists a control $\alpha(\cdot): \mathbb{R}_{\geqslant 0} \to \overline{B_m}$ such that the corresponding trajectory $x(\cdot; \alpha, x)$ tends to 0.
2. (Lyapunov stability) For each $\epsilon > 0$, there is a $\delta > 0$ such that for each $x \in \mathbb{R}^N$ with $\|x\| \leqslant \delta$ there exists a control $\alpha(\cdot): \mathbb{R}_{\geqslant 0} \to \overline{B_m}$ such that the corresponding trajectory $x(\cdot; \alpha, x)$ converges to the origin and satisfies $\|x(t; \alpha, x)\| \leqslant \epsilon$ for all $t \geqslant 0$.

Given a GAC control system of the form (1), the purpose of the stabilization problem is to study the possible existence of a feedback $\alpha(\cdot): \mathbb{R}^N \mapsto \overline{B_m}$ which makes the closed-loop system

$$\dot{x} = f(x, \alpha(x)) = \sum_{i=1}^{m} \alpha_i(x) f_i(x), \tag{2}$$

globally asymptotically stable. In the last twenty years this subject has been the focus of considerable research.[1] It is well-known that continuous stabilizing feedbacks do not exist in general; there are globally asymptotically controllable control systems which admit no continuous stabilizing feedbacks. The first example of such a system was given in 1979 by Sussmann in [33]. Then, in 1983 Brockett [7] produced a topological necessary condition which makes obstruction to the existence of such regular feedbacks; this condition provided a number of counterexamples such as the famous nonholonomic integrator. Moreover in the case of affine control systems, Artstein related the existence of continuous stabilizing feedbacks to the existence of a smooth control-Lyapunov function. This latter result showed that a GAC affine control system which does not admit a continuous stabilizing feedback cannot have a smooth control-Lyapunov function. Therefore all these results plead for the design of discontinuous stabilizing feedbacks and also for a new concept of nonsmooth control-Lyapunov function. Many authors such as, Sussmann [33], Clarke, Ledyaev, Sontag and Subbottin [10], Ancona and Bressan [3], or Rifford [21,22,24] proved the existence of discontinuous stabilizing feedback laws under general assumptions. Among them, only [10,21,22,24] made use of a nonsmooth control-Lyapunov function. In the present paper, our aim is to develop further the work which was initiated in these papers and to establish a strong link between the presence of a nonsmooth control-Lyapunov function and the construction of discontinuous stabilizing feedbacks. Moreover we also present a new kind of smooth stabilizing feedback which is of interest in the stabilization problem.

Definition 2. [2] A control-Lyapunov function (abreviated CLF) for the system (1) is a continuous function $V: \mathbb{R}^N \to \mathbb{R}$ which is positive definite, proper and such that it is a viscosity supersolution of the following Hamilton–Jacobi equation:

$$\max_{\alpha \in \overline{B_m}} \{ -\langle f(x, \alpha), DV(x) \rangle \} - V(x) \geqslant 0. \tag{3}$$

In 1983, Sontag [30] introduced the framework of nonsmooth CLF and proved the equivalence between global asymptotic controllability and the existence of a continuous control-Lyapunov function. Later, revealing the importance of semiconcavity in the design of discontinuous stabilizing feedbacks (in the spirit of [10]), we extended Sontag's Theorem and proved that every GAC control system admits a continuous CLF which is semiconcave outside the origin (see [23]). We utilized the semiconcavity property in order to construct feedbacks which make the closed-loop system globally asymptotically stable in the sense of Carathéodory. Our construction provided a simple way to design stabilizing feedback laws which were continuous on an open dense subset of the state space,

[1] We recommend to the reader the historical accounts of Coron [13] and Sontag [31].

[2] This definition takes into account the exponential decrease condition that we introduced in the framework of nonsmooth control-Lyapunov functions. Moreover we recall that the property (3) is equivalent to the following in terms of proximal subgradients

$$\forall x \in \Omega,\ \forall \zeta \in \partial_P V(x), \quad \min_{\alpha \in \overline{B_m}} \{ \langle f(x, \alpha), \zeta \rangle \} \leqslant -V(x).$$