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On self-similarity and stationary problem for fragmentation and coagulation models

Solutions auto-similaires et stationnaires pour des modèles de fragmentation et de coagulation

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Abstract

We prove the existence of a stationary solution of any given mass to the coagulation–fragmentation equation without assuming a detailed balance condition, but assuming instead that aggregation dominates fragmentation for small particles while fragmentation predominates for large particles. We also show the existence of a self-similar solution of any given mass to the coagulation equation and to the fragmentation equation for kernels satisfying a scaling property. These results are obtained, following the theory of Poincaré–Bendixson on dynamical systems, by applying the Tykonov fixed point theorem on the semigroup generated by the equation or by the associated equation written in "self-similar variables". Moreover, we show that the solutions to the fragmentation equation with initial data of a given mass behaves, as $t \to +\infty$, as the unique self similar solution of the same mass.

Résumé

Pour toute masse donnée, nous démontrons l'existence d'au moins une solution stationnaire pour l'équation de coagulation–fragmentation. Nous ne faisons pas d'hypothèse d'équilibre en détails sur les coefficients mais nous supposons que la coagulation domine la fragmentation pour les particules de petite taille et que la fragmentation est prépondérante pour les particules de grande taille. Nous démontrons également l'existence de solutions auto-similaires pour l'équation de coagulation et pour l'équation de fragmentation sous une hypothèse d'homogénéité sur les noyaux. Ces résultats sont obtenus, s'inspirant de la preuve

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du Théorème de Poincaré–Bendixson, en appliquant le théorème de point fixe de Tykonov sur le semi-groupe engendré par l'équation ou par l'équation écrite en variables auto-similaires associée. Enfin, nous démontrons que les solutions de l'équation de fragmentation de masse donnée $\rho > 0$ se comportent en temps grand comme la solutions auto-simialire de masse ρ.

MSC: 82C40; 60J75

Keywords: Equilibrium; No detailed balance condition; Poincaré–Bendixson's Theory; Tykonov fixed point theorem; Self-similar solutions; Uniqueness; Existence; Convergence to self-similarity

1. Introduction and notations

We consider the Cauchy problem for a spatially homogeneous kinetic equation modelling the dynamic of a system of particles which undergo linear (secondary) fragmentation and/or aggregation events. More precisely, if we denote by $f(t,y) \geqslant 0$ the density of particles with mass $y \in \mathbb{R}_+ := (0,\infty)$ at time $t \geqslant 0$, we study the following equation

$$\frac{\partial f}{\partial t} = Q(f) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \tag{1.1}$$

$$f(0) = f_{\text{in}} \quad \text{in } \mathbb{R}_+. \tag{1.2}$$

The coagulation–fragmentation operator Q splits into two terms

$$Q(f) = Lf + C(f). \tag{1.3}$$

The first term, Lf, describes the spontaneous fragmentation of one (mother) particle in several (possibly infinity) (daughters) particles. This process may be schematically written as

$$\{y\} \to \{y^{(1)}\} + \cdots + \{y^{(k)}\} + \cdots$$

with the mass conservation condition

$$y^{(i)} \geqslant 0, \quad y = \sum_{k=1}^{\infty} y^{(k)}.$$

The linear fragmentation operator L reads

$$Lf(y) = \int_y^\infty b(y'',y) f''\,dy'' - f(y)\int_0^y \frac{y'}{y} b(y,y')\,dy', \tag{1.4}$$

where $b = b(y,y')$ corresponds to the formation rate of particles of size y' by fragmentation of a particles of size y'. Here and below, we use the shorthand notations $\psi = \psi(y)$, $\psi' = \psi(y')$ and $\psi'' = \psi(y'')$ for any function ψ on $\mathbb{R}_+$. We will consider a fragmentation rate b satisfying

$$b(y,y') = b_0(y) B(y'/y), \tag{1.5}$$

where b_0 is a function and B is a measure such that

$$b_0(y) = y^\gamma, \quad \gamma \geqslant -1, \quad B \geqslant 0, \quad \text{with } \operatorname{supp} B \subset [0,1], \quad \int_0^1 y\,dB(y) < +\infty. \tag{1.6}$$