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Appl. Comput. Harmon. Anal. 18 (2005) 3–24
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

BMO, boundedness of affine operators, and frames

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Received 8 January 2004; revised 20 May 2004; accepted 20 May 2004

Available online 30 December 2004

Communicated by Christopher Heil (SI Guest Editor)

Abstract

This paper addresses the construction of wavelet frames as an application of the modern theory of singular integrals. The continuous wavelet inversion formula (Calderón reproducing formula) may be viewed as the action of a Calderón–Zygmund singular integral operator. Wavelet frame operators arise as Riemann sum approximations of these singular integrals. When the analyzing and synthesizing functions are smooth and have a vanishing moment, boundedness of the approximations is a simple matter of applying, for example, the Cotlar lemma. Here we investigate the situation when only one of the analyzing/synthesizing pair has a vanishing moment. The dyadic discretizations are no longer automatically bounded. We show how the $T(\mathbf{1})$ theorem may be used to find criteria under which boundedness and invertibility are ensured. Parallels between these ideas and the frame criteria of Daubechies and Ron–Shen are also discussed.

MSC: 42C15; 40A30

Keywords: Frame operator; Hardy space; BMO; Calderón–Zygmund; operator; Wavelet; Singular integral

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1 Supported by ARO contract DAAD19-02-1-0211 and by a Macquarie University MURG grant.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2004.05.005

1. Introduction and main results

Given a function ϕ on the line and real numbers $r > 1$, $s > 0$, we define the action of the dilation operator D_r and translation operator T_s on ϕ by $D_r\phi(x) = r^{1/2}\phi(rx)$, $T_s\phi(x) = \phi(x - s)$. When $r = 2$, $s = 1$, we write $D_2 = D$, $T_1 = T$. These operators play a central role in the theory of affine frames, and also in the theory of singular integral operators. The dilation and translation operators may be iterated and composed to obtain functions $\phi_{k\ell}(x) = D_r^k T_s^\ell \phi(x) = r^{k/2}\phi(r^k x - s\ell)$. The pair (r, s) generates a *mesh* $\Lambda = \Lambda^{(r,s)}$ in the upper half-plane $H = \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$ defined by $\Lambda^{(r,s)} = \{(s\ell r^{-k}, r^{-k});\ k, \ell \in \mathbb{Z}\}$.

When constructing affine frames, it is standard practice to consider the *analysis operator* A_ϕ which takes a signal $f \in L^2(\mathbb{R})$ to the doubly-indexed sequence $(A_\phi)_{k\ell}(f) = \langle f, \phi_{k\ell}\rangle = \int_{-\infty}^{\infty} f(x) r^{k/2}\overline{\phi}(r^k x - s\ell)\,\mathrm{d}x$ $(k, \ell \in \mathbb{Z})$. When $\phi \in L^2(\mathbb{R})$, these inner products are well defined, but we will further restrict attention to the case where $A_\phi : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ is bounded. In this case we may consider the adjoint operator $S_\phi = A_\phi^* : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R})$ (the *synthesis operator*), which maps a sequence $a = \{a_{k\ell}\}$ to the function $(S_\phi a)(x) = \sum_{k,\ell=-\infty}^{\infty} a_{k\ell}\phi_{k\ell}(x)$. The collection $\{\phi_{k\ell}\}_{k,\ell=-\infty}^{\infty}$ is a *frame* for $L^2(\mathbb{R})$ when A_ϕ satisfies the *frame estimates*

$$A\|f\|_{L^2(\mathbb{R})} \leqslant \|A_\phi f\|_{\ell^2(\mathbb{Z}^2)} \leqslant B\|f\|_{L^2(\mathbb{R})} \tag{1}$$

for constants $0 < A \leqslant B < \infty$. The constants A and B are the *frame bounds*. The upper bound in (1) ensures the boundedness of A_ϕ while the lower bound gives the boundedness of its inverse. In this case, the *discrete sum operator* $\mathcal{D}_\phi = S_\phi \circ A_\phi$ is bounded on $L^2(\mathbb{R})$ with bounded inverse and each $f \in L^2(\mathbb{R})$ admits the *frame expansion*

$$f = \mathcal{D}_\phi \circ \mathcal{D}_\phi^{-1} f = \sum_{k,\ell=-\infty}^{\infty} \big\langle f, \mathcal{D}_\phi^{-1} D_r^k T_s^\ell \phi\big\rangle D_r^k T_s^\ell \phi,$$

where we have used the self-adjointness of $\mathcal{D}_\phi$. The inverse $\mathcal{D}_\phi^{-1}$ is often computable via a Neumann series.

When $\phi, \psi \in L^2(\mathbb{R})$ and A_ϕ, A_ψ satisfy (1), we define

$$\mathcal{D}_{\phi\psi}^{(r,s)} f = S_\psi \circ A_\phi f = \sum_{k,\ell=-\infty}^{\infty} \big\langle f, D_r^k T_s^\ell \phi\big\rangle D_r^k T_s^\ell \psi.$$

When the mesh parameters (r, s) are fixed we will often write just $\mathcal{D}_{\phi\psi}$ for $\mathcal{D}_{\phi\psi}^{(r,s)}$. When $\mathcal{D}_{\phi\psi}$ is invertible, each $f \in L^2(\mathbb{R})$ admits the expansion

$$f = \mathcal{D}_{\phi\psi} \circ \mathcal{D}_{\phi\psi}^{-1} f = \sum_{k,\ell=-\infty}^{\infty} \big\langle f, \mathcal{D}_{\psi\phi}^{-1} D_r^k T_s^\ell \phi\big\rangle D_r^k T_s^\ell \psi, \tag{2}$$

where we have used the fact that $\mathcal{D}_{\phi\psi}^* = \mathcal{D}_{\psi\phi}$. The expansion (2) remains valid when the roles of ϕ and ψ are interchanged. This approach, requiring as it does the boundedness of A_ϕ and A_ψ, requires both ϕ and ψ to have a vanishing moment, i.e., $\int \phi = \int \psi = 0$ when φ and ψ are also integrable.

The approach taken in this paper is slightly different. We focus on the composition $\mathcal{D}_{\phi\psi} = S_\psi \circ A_\phi$ rather than the individual operators A_ϕ and S_ψ. The theory of singular integral operators and *operators of Cotlar type* can be brought to bear on the analysis of $\mathcal{D}_{\phi\psi}$ from which norm estimates are explicitly calculable. It is not our purpose here to review the role of Calderón–Zygmund theory in the analysis