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Symmetric MRA tight wavelet frames with three generators and high vanishing moments

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Abstract

Let ϕ be a compactly supported symmetric real-valued refinable function in $L_2(\mathbb{R})$ with a finitely supported symmetric real-valued mask on $\mathbb{Z}$. Under the assumption that the shifts of ϕ are stable, in this paper we prove that one can always construct three wavelet functions ψ^1, ψ^2, and ψ^3 such that

(i) All the wavelet functions ψ^1, ψ^2, and ψ^3 are compactly supported, real-valued and finite linear combinations of the functions $\phi(2\cdot -k)$, $k \in \mathbb{Z}$;
(ii) Each of the wavelet functions ψ^1, ψ^2, and ψ^3 is either symmetric or antisymmetric;
(iii) $\{\psi^1, \psi^2, \psi^3\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$, that is,

$$\|f\|^2 = \sum_{\ell=1}^{3} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\langle f, \psi_{j,k}^{\ell}\rangle\right|^2 \quad \forall f \in L_2(\mathbb{R}),$$

where $\psi_{j,k}^{\ell} := 2^{j/2}\psi^{\ell}(2^j \cdot -k)$, $\ell = 1, 2, 3$ and $j, k \in \mathbb{Z}$;
(iv) Each of the wavelet functions ψ^1, ψ^2, and ψ^3 has the highest possible order of vanishing moments, that is, its order of vanishing moments matches the order of the approximation order provided by the refinable function ϕ.

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We shall give an example to demonstrate that the assumption on stability of the refinable function ϕ cannot be dropped. Some examples of symmetric tight wavelet frames with three compactly supported real-valued symmetric/antisymmetric generators will be given to illustrate the results and construction in this paper.

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1. Introduction and motivation

Orthonormal wavelets and their various generalizations have been extensively studied in the literature and they have been successfully applied to many applications such as image processing and signal denoising [1–24]. In this paper, we are particularly interested in tight wavelet frames that are derived from refinable functions via a multiresolution analysis. A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. Tight wavelet frames have some desirable features, such as near translation invariant wavelet frame transforms, and it may be easier to recognize patterns in a redundant transform. For advantages and applications of tight wavelet frames, the reader is referred to [1–5,8–15,18,20–24] and many references therein.

Before proceeding further, let us recall some basic definitions. We say that a set $\{\psi^1, \ldots, \psi^r\}$ of functions in $L_2(\mathbb{R})$ generates a (normalized) *tight wavelet frame* in $L_2(\mathbb{R})$ if

$$\|f\|^2 = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left|\langle f, \psi^\ell_{j,k}\rangle\right|^2 \quad \forall f \in L_2(\mathbb{R}), \tag{1.1}$$

with $\psi^\ell_{j,k} := 2^{j/2}\psi^\ell(2^j \cdot -k)$, and where $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$ and $\|f\|^2 := \langle f, f\rangle$. The set $\{\psi^1, \ldots, \psi^r\}$ is called a set of generators for the corresponding tight wavelet frame. Let δ denote the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Suppose that $\{\psi^1, \ldots, \psi^r\}$ generates a tight wavelet frame in $L_2(\mathbb{R})$. Then $\{\psi^1, \ldots, \psi^r\}$ generates an orthonormal wavelet basis in $L_2(\mathbb{R})$ if and only if $\langle \psi^\ell_{j,k}, \psi^{\ell'}_{j',k'}\rangle = \delta_{\ell-\ell'}\delta_{j-j'}\delta_{k-k'}$ for all $\ell, \ell' = 1, \ldots, r$ and $j, j', k, k' \in \mathbb{Z}$ (or, if and only if $\|\psi^1\| = \cdots = \|\psi^r\| = 1$). It follows directly from (1.1) that any function $f \in L_2(\mathbb{R})$ has the wavelet expansion

$$f = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j,k}\rangle \psi^\ell_{j,k}.$$

In order to have a fast wavelet frame transform, tight wavelet frames are generally derived from refinable functions via a multiresolution analysis. A function ϕ is called a *refinable function* if it satisfies the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} a_k \phi(2 \cdot -k), \tag{1.2}$$