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Banach frames in coorbit spaces consisting of elements which are invariant under symmetry groups

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Abstract

This paper is concerned with the construction of atomic decompositions and Banach frames for subspaces of certain Banach spaces consisting of elements which are invariant under some symmetry group. These Banach spaces—called coorbit spaces—are related to an integrable group representation. The construction is established via a generalization of the well-established Feichtinger–Gröchenig theory. Examples include radial wavelet-like atomic decompositions and frames for radial Besov–Triebel–Lizorkin spaces, as well as radial Gabor frames and atomic decompositions for radial modulation spaces.

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1. Introduction

The study of time–frequency analysis and wavelet analysis of functions on $\mathbb{R}^d$ that are invariant under a symmetry group was started in [17]. There the author raised the question whether it is possible to exploit

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the symmetry in order to reduce complexity, improve approximation quality, etc., in Gabor or wavelet analysis.

Imagine that a function f on $\mathbb{R}^d$, which has some symmetries, is represented by a Gabor or wavelet expansion. Then the functions (translates and dilates or modulations of a single function) in the expansion will not all (actually nearly none of them) obey the same symmetry properties as f. So one might ask whether it is possible to find a Gabor-like frame or wavelet-like frame (for the subspace of $L^2(\mathbb{R}^d)$ consisting of invariant functions) such that each frame element itself is invariant under the symmetry group.

In case of radial symmetry in $\mathbb{R}^d$, Epperson and Frazier successfully constructed radial wavelet frames which even serve as atomic decompositions for subspaces of Besov spaces and Triebel–Lizorkin spaces consisting of radial functions [5]. Kühn et al. used this radial atomic decomposition to establish results concerning compact embeddings of radial Besov spaces in [15]. In dimension 3 radial orthonormal wavelets were constructed in [19] using the concept of a multiresolution analysis. However, concerning radial Gabor frames there seems nothing to be known up to now.

Both wavelet theory and time–frequency analysis can be treated simultaneously using representation theory of locally compact groups. In this abstract setting the theory for the continuous transform in the presence of invariance under a general symmetry group was developed in [17]. The symmetry group is realized as compact automorphism group of the locally compact group whose representation coefficients generate the continuous transform. As examples, the continuous wavelet transform and the short time Fourier transform (STFT) of radial functions on $\mathbb{R}^d$ were discussed in detail. A radial function can be described by some function on the positive halfline $\mathbb{R}_+$ and it turned out in [17] that the continuous wavelet transform and the STFT of a radial function can be computed by an integral transform on $\mathbb{R}_+$, which involves a generalized translation in case of the wavelet transform and some kind of a generalized combined translation and modulation (formula (4.4) in [17]) in case of the STFT. Both of these "generalized operations" are given as integrals and in particular the generalized combined translation/modulation turns out to be quite complicated.

The (stable) discretization of the "radial wavelet transform" and the "radial STFT" actually means the construction of frames, where each frame element is given as some generalized translation or as some generalized translation/modulation of a single function. In order to attack the discretization problem, the first idea would probably be to proceed analogously to the classical wavelet and Gabor theory. And in fact, in case of radial wavelets in $\mathbb{R}^3$ this approach was successful [19]. However, in arbitrary dimension and for radial Gabor frames the direct approach seems hopeless because of the complicated form of the combined generalized translation/modulation. So one has to look for different approaches.

In the classical setting (i.e., without symmetry group) Feichtinger–Gröchenig theory has proven to provide a general and very flexible way to construct coherent atomic decompositions and Banach frames for certain Banach spaces, called coorbit spaces, which are related to the continuous transform [8–10,12]. This approach makes heavy use of group theory and, thus, is quite abstract. However, the final outcome is a very elegant solution to the discretization problem. In particular, regular and irregular Gabor and wavelet frames are included as examples. Moreover, not only Hilbert space theory is covered but also atomic decompositions and Banach frames of Besov–Triebel–Lizorkin spaces and of modulation spaces are provided. So it also provides a new aspect of the theory of function spaces.

Motivated by its success, it seemed very promising to attack the problem of constructing frames, where each frame element is invariant under some symmetry group, by generalizing the Feichtinger–Gröchenig theory. And in fact, this paper presents the results of this approach. As in [8–10,12] we make use of