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Connectivity in the set of Gabor frames

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Abstract

In this paper we present a constructive proof that the set of Gabor frames is path-connected in the $L^2(\mathbb{R}^n)$-norm. In particular, this result holds for the set of Gabor–Parseval frames as well as for the set of Gabor orthonormal bases. In order to prove this result, we introduce a construction which shows exactly how to modify a Gabor frame or Parseval frame to obtain a new one with the same property. Our technique is a modification of a method used in [Glas. Mat. 38 (58) (2003) 75–98] to study the connectivity of affine Parseval frames.

MSC: 42C15; 42C40

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1. Introduction

The study of the topological properties of Gabor and affine systems is an important topic in the wavelet and Gabor theory. Some special prominence has been given in the literature to the problem of connectivity. This question was originally raised in [1], where the significance of the problem is also emphasized. Despite several important contributions to the study of this problem (for example in [4,5,12,13]) there are still a number of open questions.

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In this paper, we are concerned with the problem of the connectivity for Gabor systems. In [3,4], Gabardo, Han, and Larson used an abstract result from the theory of von Neumann algebras to prove that the sets of Gabor–Parseval frames and Gabor frames are path-connected in the norm topology of L^2. Unfortunately, their proof is not constructive and this is a limitation since, in many situations, one would like to explicitly construct the path connecting any Gabor–Parseval frame or frame to a fixed element in the same set. The main contribution of this paper is to present a *constructive* proof of these results. Unlike the abstract Hilbert space method in [3,4], our approach involves an explicit deformation g_t, $t \in [0,1]$, of an arbitrary frame generator g_0, connecting g_0 to a fixed band-limited generator g_1. If $g_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, such deformation is continuous in L^p, $1 \leqslant p < \infty$. Furthermore, we obtain an explicitly controlled deformation of the Gabor frame coefficients by uniformly continuous functions. The techniques developed in this paper are relevant to different problems, including the study of the stability of frames under perturbation.

To illustrate our technique, consider the one-dimensional Gabor systems

$$\mathcal{G}_b(g) = \left\{e^{2\pi i b m x} g(x-k) \colon k, m \in \mathbb{Z}\right\},$$

where $g \in L^2(\mathbb{R})$, $0 < b < 1$. For $E = [0, b)$, let $g_1 = (\chi_E)^\vee$. Then the system $\mathcal{G}_b(g_1)$ is a Parseval frame for $L^2(\mathbb{R})$. In our approach, we connect any g_0 such that $\mathcal{G}_b(g_0)$ is a Parseval frame for $L^2(\mathbb{R})$ to g_1 in the following way. For $0 \leqslant t \leqslant 1$, let $E_t = [0, bt)$ and define a deformation g_t by

$$\hat{g}_t(\xi) = \begin{cases} \hat{g}_0(\xi), & \xi \in \hat{\mathbb{R}} \setminus \tau_b(E_t), \\ 1, & \xi \in E_t, \end{cases}$$

where $\tau_b(E_t) = \bigcup_{k \in \mathbb{Z}} (E_t + bk)$. As we mentioned, we can show that this deformation is continuous in L^p, $1 \leqslant p < \infty$.

Before describing our approach in details, it will be useful to establish some notation and definitions.

1.1. Preliminaries

In this paper, $GL_n(\mathbb{R})$ denotes the $n \times n$ invertible matrices with real coefficients and, similarly, $GL_n(\mathbb{Q})$ denotes the $n \times n$ invertible matrices with rational coefficients. The Fourier transform is defined as $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, \mathrm{d}x$ and the inverse Fourier transform is $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi}\, \mathrm{d}\xi$. Throughout the paper, the space $\mathbb{T}^n$ will be identified with $[0,1)^n$. The Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$ is denoted by $\mu(\Omega)$.

Λ is a *lattice* in $\mathbb{R}^n$ if $\Lambda = A\mathbb{Z}^n$, where $A \in GL_n(\mathbb{R})$. Given a measurable set $\Omega \subseteq \mathbb{R}^n$ and a lattice Λ in $\mathbb{R}^n$, we say that Ω *tiles* $\mathbb{R}^n$ by Λ, or Ω is a *fundamental domain* of Λ, if the following two properties hold:

(i) $\bigcup_{\ell \in \Lambda} (\Omega + \ell) = \mathbb{R}^n$ a.e.;
(ii) $\mu((\Omega + \ell) \cap (\Omega + \ell')) = 0$ for any $\ell \neq \ell'$ in Λ.

We say that Ω *packs* $\mathbb{R}^n$ by Λ if only (ii) holds. Equivalently, Ω tiles $\mathbb{R}^n$ by Λ if and only if

$$\sum_{\ell \in \Lambda} \chi_\Omega(x - \ell) = 1 \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.1}$$