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Letter to the Editor

A new friendly method of computing prolate spheroidal wave functions and wavelets

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Abstract

Prolate spheroidal wave functions, because of their many remarkable properties leading to new applications, have recently experienced an upsurge of interest. They may be defined as eigenfunctions of either a differential operator or an integral operator (as observed by Slepian in the 1960s). There are various ways of calculating their values based on both approaches. The standard one uses an approximation based on Legendre polynomials, which, however, is valid only on a finite interval. An alternative, valid in a neighborhood of infinity, uses a Bessel function approximation. In this letter we present a new method based on an eigenvalue problem for a matrix operator equivalent to that of the integral operator. Its solution gives the values of these functions on the entire real line and is computationally more efficient.

Keywords: Sampling theory; Prolate spheroidal wave functions

1. Introduction

In this work we shall be concerned with the construction of prolate spheroidal wave functions (PSWFs) and their associated prolate spheroidal wavelets (PS wavelets). The former were introduced in a classic paper [7] by David Slepian and his collaborators in Bell Labs as solutions of an energy concentration problem. They had previously been known as solutions of a Sturm–Liouville problem, from which many

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of their properties could be derived. The scaling function of the PS wavelets introduced in [13] was based on the first PSWF.

Both sets of functions have many interesting, even unique, properties that make them desirable as bases [12,13]. Some of these properties are listed in the next section, and will be used to convert the energy concentration problem from one involving integrals to one involving sequences. This in turn will enable us to construct the PSWFs and the PS scaling function from a discrete eigenvalue problem. It should be pointed out that this discrete problem is not the one arising from some standard numerical methods, but rather is unique to this setting and gives *exactly* the same eigenvalues as the continuous integral equation. These eigenvalues have the surprising "step function property": they are very close to 1 for $n < \pi\tau$, and very close to 0 for larger values of n [7].

After Slepian, Pollak, and Landau discovered the connection between PSWFs and the energy concentration problem during the 1960s, the PSWFs were shown to be an important tool for analyzing some problems raised in signal processing and telecommunications [5]. But they did not have a standard representation in terms of trigonometric functions and were, and still are, regarded as somewhat mysterious. They are seldom used in practice because of this and because the computation of the PSWF function values themselves is a complex numerical problem [4]. Most of the standard methods of computing PSWFs involve an expansion in Legendre polynomials for small values of t and expansion in Bessel functions for large values [11]. In practice, it is often more convenient to use published tabulated values [1,2] to construct the PSWFs, but then one is restricted to the values of the parameters in the tables. Although some computer programs for evaluating the PSWFs are available [10,14], many are not portable, or not have been tested thoroughly. Our method can be easily programmed in MAPLE, it holds for all values of t simultaneously, it is easily extended to higher dimensions, and does not involve calculating integrals. It should make this useful tool more widely accessible to both researches and students.

2. Background

The prolate spheroidal wave functions, (PSWFs) $\{\varphi_{n,\sigma,\tau}(t)\}$, constitute an orthonormal basis of the space of σ-bandlimited functions on the real line, i.e., functions whose Fourier transforms have support on the interval $[-\sigma,\sigma]$. The PSWFs are maximally concentrated on an interval $[-\tau,\tau]$ in a sense described below and depend on parameters σ and τ. There are several ways of characterizing them:

- as the eigenfunctions of a differential operator arising from a Helmholz equation on a prolate spheroid

$$\left(\tau^2 - t^2\right)\frac{\mathrm{d}^2\varphi_{n,\sigma,\tau}}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}\varphi_{n,\sigma,\tau}}{\mathrm{d}t} - \sigma^2 t^2\varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau}\varphi_{n,\sigma,\tau};$$

- as the maximum energy concentration of a σ-bandlimited function on the interval $[-\tau,\tau]$; that is, $\varphi_{0,\sigma,\tau}$ is the function of total energy 1 $(=\|\varphi_{0,\sigma,\tau}\|^2)$ such that

$$\int_{-\tau}^{\tau}\left|f(t)\right|^2\,\mathrm{d}t$$

  is maximized, $\varphi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\varphi_{0,\sigma,\tau}$, etc.; or