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The infimum cosine angle between two finitely generated shift-invariant spaces and its applications ☆

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Abstract

An expression of the infimum cosine angle between two finite dimensional subspaces is given in terms of Gramians and is applied to find a useful formula for the infimum cosine angle between two finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ in terms of the generators. We then present new equivalent conditions for the existence of the oblique projection between two finitely generated shift-invariant subspaces, thereby providing a constructive method to generate oblique dual frames. This result is a generalization of a result of Aldroubi and is closely related with the biorthogonality of two frame multiresolution analyses. Finally, we illustrate our results by examples.

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1. Introduction

Throughout this article $\mathcal{H}$ denotes a separable Hilbert space over the complex field $\mathbb{C}$. The purpose of this article is to analyze the concept of the *infimum cosine angle* $R(U, V)$ between two closed subspaces U and V of $\mathcal{H}$ which is defined as follows [1,42]:

$$R(U, V) := \inf_{u \in U \setminus \{0\}} \frac{\|P_V u\|}{\|u\|},$$

where P_V denotes the orthogonal projection onto V. The arc-cosine value of $R(U, V)$ is usually interpreted as the 'largest angle' between U and V [42]. If we take the supremum, instead of the infimum, of the right-hand side of the above equation we have the so-called *supremum cosine angle* $S(U, V)$ of U and V, and the two angles are related by the following relation: $R(U, V) = (1 - S(U, V^{\perp})^2)^{1/2}$ [42]. Similar to $R(U, V)$, the arc-cosine value of $S(U, V)$ is interpreted as the 'smallest angle' between U and V [42]. We use the convention that $R(U, V) = 1$ if U is trivial for the obvious reason. Note also that if U is not trivial and V is trivial, then $R(U, V) = 0$. See [1,42] for the geometric meaning of this concept and its applications to signal processing, and see [2,9,10,29,30] for its applications to the theory of wavelets. Even though $R(U, V) = R(V^{\perp}, U^{\perp})$, $R(U, V) \neq R(V, U)$ in general, whereas $S(U, V) = S(V, U)$ [9,42]. As will be mentioned, $R(U, V)$ is closely related with the biorthogonality of two multiresolution analyses, and the perturbation of frames in shift-invariant subspaces. In this article we concentrate on the infimum cosine angle, and postpone the discussion of the supremum cosine angle to the forthcoming paper [32], in which the connection between $S(U, V)$ and the closedness of the sum $U + V$ is analyzed [32].

We now explain the motivation for investigating the infimum cosine angle. First, the infimum cosine angle between two finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ is closely related with the biorthogonality of two multiresolution analyses [1,2,9,29,30,41]. The infimum cosine angle, however, in the cited papers is considered under various restrictive conditions on the generating sets of the shift-invariant subspaces. See Section 4 for the definition of the shift-invariant subspace of $L^2(\mathbb{R}^d)$. More specifically, the authors in the cited papers consider either the case where the shifts (i.e., (multi-)integer translates) of the multiple generating sets form Riesz bases for the shift-invariant spaces and the cardinalities of the multiple generating sets coincide [1,9,41] or the case where the generating sets are singletons [2,29,30,43]. Therefore, the results in the existing literature are insufficient to deal with the case where the shifts of the multiple generating sets form frames for the shift-invariant spaces or the one where the cardinalities of the generating sets are different. Notice that the latter cases occur if we consider frame multiresolution analyses [3,4,14,18,20,22,30,31,33,34,36,37]. In this article, we consider the infimum cosine angle between two finitely generated shift-invariant subspaces under no assumption on the generating sets (Theorem 4.7). Therefore, our results can be applied to the more general form of the biorthogonal multiresolution analyses.

Second, we mention that the connection between the infimum cosine angle and the perturbation of frames in shift-invariant subspaces will be discussed in a recent paper by Christensen and co-authors [16].

Even though many of our results in this article can be generalized to infinitely generated shift-invariant subspaces, we restrict our attention to finitely generated shift-invariant spaces for the following reasons. In the conventional theory of multiresolution analysis, the central space is a finitely generated shift-invariant space. Moreover, if the central space is not regular (see Section 4 for the definition), then there