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Multiscale approximation of piecewise smooth two-dimensional functions using normal triangulated meshes

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Abstract

Multiresolution triangulation meshes are widely used in computer graphics for representing three-dimensional (3-d) shapes. We propose to use these tools to represent 2-d piecewise smooth functions such as grayscale images, because triangles have potential to more efficiently approximate the discontinuities between the smooth pieces than other standard tools like wavelets. We show that *normal mesh subdivision* is an efficient triangulation, thanks to its local *adaptivity* to the discontinuities. Indeed, we prove that, within a certain function class, the normal mesh representation has an optimal asymptotic error decay rate as the number of terms in the representation grows. This function class is the so-called *horizon class* comprising constant regions separated by smooth discontinuities, where the line of discontinuity is C^2 continuous. This optimal decay rate is possible because normal meshes automatically generate a polyline (piecewise linear) approximation of each discontinuity, unlike the blocky piecewise constant approximation of tensor product wavelets. In this way, the proposed nonlinear multiscale normal mesh decomposition is an anisotropic representation of the 2-d function. The same idea of anisotropic representations lies at the basis of decompositions such as wedgelet and curvelet transforms, but the proposed normal mesh approach has a unique construction.

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1. Introduction: images with long smooth edges

This paper concerns the representation and approximation of piecewise smooth, two-dimensional (2-d) functions, which consist of smooth regions delineated by step discontinuities along smooth one-dimensional (1-d) contours, which we call *edges*. Many different types of real-world data can be modeled as piecewise smooth. As an important example, a piecewise smooth function is a quite accurate model for a grayscale *image*, which represents the light intensity of a black-and-white visual scene. While we will use images as our central, running example in this paper, other examples abound in statistics and differential equations for a broad spectrum of applications.

By *approximation*, we mean approximating a piecewise smooth function with a finite-dimensional representation. Immediate applications of approximation results include compression and noise removal (denoising).

For images and many other kinds of data, an approximation is typically defined on a discrete set of points on some grid. For example, digital images are typically acquired by sampling the light intensity at discrete points on a square grid of pixels (currently using a CCD array), and so image representations and processing algorithms typically operate on this square grid. The square pixel grid is nearly always assumed to be fixed, with the dependent variable of the image the pixel intensity. While the acquisition and processing of image data on a square grid of pixels is simple, it turns out to be very inefficient for representing many important image features, including the *edges*.

Edges are the dominating features in piecewise smooth 2-d functions. Edges contain two types of information: *where* the edge is located, i.e., its location and geometry, and *what* is the step value, i.e., the height of the discontinuity. In 2-d, geometry information plays a crucial role, much more than in 1-d. In 1-d piecewise smooth functions, discontinuities occur at isolated points, and these can be easily captured in a wavelet transform. In 2-d, edge singularities lie along 1-d contours, which are much harder to capture.

The time–scale analysis of the wavelet representation provides a powerful tool for approximating a 1-d function f. Thanks to the local support of the basis functions, under mild conditions, a nonlinear wavelet approximation f_n containing the n largest terms of the wavelet expansion of f performs as well on a piecewise smooth f as on a smooth f [7,8,13,19,20]. Indeed, the L_2 approximation error decays rapidly with increasing n:

$$\left\| f - f_n^{\text{1-d wavelet}} \right\| = \mathcal{O}\left(n^{-\nu}\right). \tag{1}$$

In this equation, ν stands for

$$\nu = \min(\tilde{p}, \alpha),$$

with $\tilde{p}$ the number of (dual) vanishing moments of the wavelet analysis and α the Lipschitz regularity of the signal at its nonsingular points. Wavelets provide a very efficient representation of 1-d piecewise smooth signals primarily because in 1-d the geometry information consists of merely a few isolated points.