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Decomposition and reconstruction of multidimensional signals using polyharmonic pre-wavelets

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Abstract

In this paper, we build a multidimensional wavelet decomposition based on polyharmonic B-splines. The pre-wavelets are polyharmonic splines and so not tensor products of univariate wavelets. Explicit construction of the filters specified by the classical dyadic scaling relations is given and the decay of the functions and the filters is shown. We then design the decomposition/recomposition algorithm by means of downsampling/upsampling and convolution products.

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1. Introduction

The theory and practice of wavelet decomposition of signals and functions is a particularly attractive research area in approximation theory and in signal processing. Their applications range from transmitting or filtering signals to the numerical solution of partial differential equations (see, e.g., [4,8]). The most used wavelets bases for multiresolution analysis (MRA) in multiple dimension are obtained through tensor products of one-dimensional functions (see, e.g., [5,6]). Quite from the beginning, a more general multi-dimensional approach was given by the pioneers of the MRA (see [15,16,21]); for the actual construction of multivariate pre-wavelets, see [7].

In this context, polyharmonic functions have been considered very often in literature. Indeed, the possibility of doing MRA generated by classes of polyharmonic splines have been studied in some detail as far back as the works [15] and [17], where the scaling function generates orthogonal Riesz basis, and many others authors continued this development [14,16,18,20]. In these works, in particular we find the class of the so-called polyharmonic B-splines. As is well known, in [22,23] polyharmonic B-splines were introduced as a finite linear combination of translates—actually a discretization of the iterated Laplacean operator Δ^m—of the fundamental solution of Δ^m. These basis functions were earlier considered in [10], then in [11] and [12] for improving the condition number of linear systems involved for thin plate spline interpolation, and were used for cardinal interpolation (see, e.g., [3]). Despite the fact that the polyharmonic B-splines violate the rapid-decay requirements of classical wavelet theory (they typically algebraically decay), they generate Riesz bases and are perfectly scaling functions for $m > s/2$. This was proved for instance by Micchelli et al. [20] for a wider class of refinable functions, or by Madych [18], cf. pp. 274–276, with a different approach.

As it appears from the cited works, the theory of polyharmonic MRA have been well designed; whereas, the aspects connected with the actual construction of the related wavelet decomposition has not been addressed all that often in the applied literature, especially in dimension greater than two. For instance, one can find a numerical application in [19] where the refinement equation of the Lagrangean polyharmonic splines is used to recover a surface by means of the Fourier transform and the usual discrete convolution product; and very recently (actually later than, but independently from this work), Van De Vill et al. [26] defined a particular polyharmonic B-spline, which they called “isotropic polyharmonic B-spline” to build a specific bi-dimensional MRA, in order to process a signal.

The aim of this paper is to provide a very explicit construction of a polyharmonic pre-wavelet decomposition, which gives possible direct implementation of the involved filters in all dimensions; the scaling function is in the class of polyharmonic B-splines with centers over the lattice Z^s and the pre-wavelets are polyharmonic splines with centers over the fine lattice $2^{-1}Z^s$. In addition, we clarify some theoretical features, specific to this decomposition, such as the rate of decay of the filters. Explicit formulae for deriving the filters and functions involved in the polyharmonic B-spline wavelet transform are here provided. Our scheme works in the spatial domain. The filters are computable once for all, getting simple procedures for code’s implementation. The wavelet decomposition/recomposition algorithm results computationally efficient since formulae involve upsampling/downsampling and convolutions, exactly as in the one-dimensional case. This offers the reader an easy use of the given wavelet decomposition, which may be useful in the applications [1,2]; in this sense this work improves the related earlier works.

The paper is organized as follows. In Section 2 we give the definitions and some properties. We define some classes of functions and vectors, which are absolutely bounded by a radial algebraically decaying function or vector, respectively, and we prove some of their properties that we use later on. In particular,