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On transitive Lie bialgebroids and Poisson groupoids ☆

Z. Chen *, Z.-J. Liu

Department of Mathematics and LMAM, Peking University, Beijing 100871, China

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Abstract

We prove that, for any transitive Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, the differential associated to the Lie algebroid structure on $\mathcal{A}^*$ has the form $d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega$, where Λ is a section of $\wedge^2\mathcal{A}$ and Ω is a Lie algebroid 1-cocycle for the adjoint representation of $\mathcal{A}$. Globally, for any transitive Poisson groupoid (Γ, Π), the Poisson structure has the form $\Pi = \overleftarrow{\Lambda} - \overrightarrow{\Lambda} + \Pi_{\mathcal{F}}$, where $\Pi_{\mathcal{F}}$ is a bivector field on Γ associated to a Lie groupoid 1-cocycle.

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1. Introduction

The notion of a Lie bialgebroid was introduced by Mackenzie and Xu in [9] as a natural generalization of that of a Lie bialgebra, as well as the infinitesimal version of Poisson groupoids introduced by Weinstein [11]. It has been shown that much of the theory of Poisson groups and Lie bialgebras can be similarly carried out in this general context. It is therefore a basic task to study the structure of Lie

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* Corresponding author.
E-mail addresses: chenzhuo@mail.cnu.edu.cn (Z. Chen), liuzj@pku.edu.cn (Z.-J. Liu).

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bialgebroids. In particular, it is very interesting to figure out what special features a Lie bialgebroid, in which the Lie algebroid structure is transitive, would have.

The purpose of this paper is to classify all Lie bialgebroids $(\mathcal{A}, \mathcal{A}^*)$, where $\mathcal{A}$ is transitive. We show that in this case the Lie bialgebroid structure can be characterized by a pair (Λ, Ω), where Λ is a section of $\wedge^2\mathcal{A}$ and Ω is a Lie algebroid 1-cocycle for the adjoint representation of $\mathcal{A}$, which defines the Lie algebroid structure on $\mathcal{A}^*$. Since any transitive Lie algebroid is locally isomorphic to one of the form $\mathcal{A} = TM \oplus (M \times \mathfrak{g})$, this result generalizes the result in [6] where such local structures are classified.

It is well known that a Lie bialgebra can be integrated to a Poisson Lie group by lifting a Lie algebra 1-cocycle. A Lie bialgebroid can also be integrated to a Poisson groupoid under certain assumptions [10]. But the integration procedure is more complicated. However, if the Lie bialgebroid is transitive, the Poisson structure on the groupoid can be easily described, as we shall see below. In this case, to integrate a Lie bialgebroid is essentially to integrate a Lie algebroid 1-cocycle to a Lie groupoid 1-cocycle as in the case of a Lie algebra.

The paper is organized as follows. In Section 2, we recall some basic definitions and known results which will be used below. In Section 3, we prove that, for any transitive Lie bialgebroid $(\mathcal{A}, \mathcal{A}^*)$, the differential associated to the Lie algebroid structure on $\mathcal{A}^*$ always has the form $d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega$, where Λ is a section of $\wedge^2\mathcal{A}$ and Ω is a Lie algebroid 1-cocycle for the adjoint representation of $\mathcal{A}$. In Section 4, we prove that, for any transitive Poisson groupoid (Γ, Π), the Poisson structure has the form $\Pi = \overleftarrow{\Lambda} - \overrightarrow{\Lambda} + \Pi_{\mathcal{F}}$, where $\Pi_{\mathcal{F}}$ is a bivector field on Γ associated to a Lie groupoid 1-cocycle $\mathcal{F}$. When Γ is α-simply connected, $\mathcal{F}$ can be obtained by integrating a Lie algebroid 1-cocycle. In Section 5, we first discuss some properties of the cohomology for Lie algebroids, which we then further study in the case of a coboundary Lie bialgebroid.

2. Preliminaries

Throughout the paper we suppose that the base of the Lie algebroids and Lie groupoids under consideration are connected. By $(\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \rho)$, we denote a Lie algebroid $\mathcal{A}$ with Lie bracket $[\cdot,\cdot]_{\mathcal{A}}$ on $\Gamma(\mathcal{A})$ and anchor map $\rho : \mathcal{A} \to TM$. By $(\Gamma \rightrightarrows M;\ \alpha, \beta)$, we denote a Lie groupoid Γ with source and target maps $\alpha, \beta : \Gamma \to M$ and we denote by

$$\Gamma^{[2]} \triangleq \big\{(p,q) \in \Gamma \times \Gamma \mid \beta(p) = \alpha(q)\big\}$$

the set of composable pairs of points.

Lie bialgebroids and Poisson groupoids. A Lie bialgebroid is a pair of Lie algebroids $(\mathcal{A}, \mathcal{A}^*)$ satisfying the following compatibility condition

$$d_*[u,v]_{\mathcal{A}} = [d_*u, v]_{\mathcal{A}} + [u, d_*v]_{\mathcal{A}}, \quad \forall u, v \in \Gamma(\mathcal{A}), \tag{1}$$

where the differential d_* on $\Gamma(\wedge^\bullet\mathcal{A})$ comes from the Lie algebroid structure on $\mathcal{A}^*$ (see [4,9] for more details). Of course, one can also denote a Lie bialgebroid by the pair $(\mathcal{A}, d_*)$, since the anchor $\rho_* : \mathcal{A}^* \to TM$ and the Lie bracket $[\cdot,\cdot]_*$ on the dual bundle are defined by d_* as follows: $\rho_*^*(df) = d_* f, \forall f \in C^\infty(M)$ and for all $u \in \Gamma(\mathcal{A}), \xi, \eta \in \Gamma(\mathcal{A}^*)$,

$$\big\langle [\xi,\eta]_*, u\big\rangle = \rho_*(\xi)\langle \eta, u\rangle - \rho_*(\xi)\langle \eta, u\rangle - d_*u(\xi,\eta).$$