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Integral geometry and geometric inequalities in hyperbolic space ☆

Eduardo Gallego, Gil Solanes *,1

Departament de Matemàtiques, Campus de la UAB, Edifici Cc, 08193 Bellaterra, Spain

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Abstract

Using results from integral geometry, we find inequalities involving mean curvature integrals of convex hypersurfaces in hyperbolic space. Such inequalities generalize the Minkowski formulas for euclidean convex sets.
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1. Introduction and results

In hyperbolic space the following isoperimetric-like inequality is well known (cf. [1])

$$\operatorname{vol}(\partial Q) > (n-1)\operatorname{vol}(Q) \tag{1}$$

for any convex domain $Q \subset \mathbb{H}^n$. This shows a strong contrast with euclidean geometry where these two volumes cannot be *linearly* compared, since for instance they are affected differently by homothetical transformations of Q. Indeed, the isoperimetric inequality in euclidean space is

$$\big(\operatorname{vol}(\partial Q)\big)^n \geqslant c\big(\operatorname{vol}(Q)\big)^{n-1} \tag{2}$$

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* Corresponding author.

E-mail addresses: egallego@mat.uab.es (E. Gallego), solanes@topolog.u-bourgogne.fr (G. Solanes).

1 Present address: Institut de Mathématiques de Bourgogne, 9 Avénue Alain Savary, BP 47870, 21078 Dijon Cedex, France.

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for a constant c. More generally, the Minkowski inequalities for euclidean convex domains Q take the form

$$\big(W_r(Q)\big)^s > c\big(W_s(Q)\big)^r, \quad r > s, \tag{3}$$

where W_i are the so-called *Quermassintegrale* whose definition is recalled a few lines below. Again, the exponents correct the different dimensions of the magnitudes, and again this will not be necessary in hyperbolic space. Indeed, the aim of this paper is to generalize (1) by finding *linear* geometric inequalities for convex domains in hyperbolic space analogue to (3). Before, let us recall how the Quermassintegrale of an euclidean convex domain are defined in the frame of integral geometry

$$W_r(Q) = \frac{(n-r)\cdot O_{n-1}}{n\cdot O_{n-r-1}\cdot \mathrm{vol}(G(n-r,n))}\int_{G(n-r,n)} \mathrm{vol}_{n-r}\big(\pi_V(Q)\big)\,\mathrm{d}V,$$

where π_V is the orthogonal projection onto the $(n-r)$-dimensional linear subspace V, and $\mathrm{d}V$ is the natural (invariant) measure in the Grassmannian of such subspaces. Here and in the following $O_i = \mathrm{vol}(\mathbb{S}^i)$. Alternatively, the Quermassintegrale are, up to constants, the measure of the set of affine subspaces intersecting the convex body (cf. [2]). Namely,

$$W_r(Q) = \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r} \chi(L\cap Q)\,\mathrm{d}L, \tag{4}$$

where $\mathcal{L}_r$ is the space of r-dimensional affine subspaces L, endowed with its natural (invariant) measure $\mathrm{d}L$. Here and in the following the function χ is just given by $\chi(Q) = 1$ whenever $Q \neq \emptyset$, and $\chi(\emptyset) = 0$. In case that ∂Q is C^2-differentiable, the Quermassintegrale coincide with the *total mean curvatures* of the boundary

$$W_r(Q) = nM_r(\partial Q) := n\int_{\partial Q} \sigma_r(x)\,\mathrm{d}x,$$

where σ_r and $\mathrm{d}x$ are respectively the rth mean curvature and the volume element of ∂Q.

Therefore, in order to generalize (3), the first point is to clarify the notion of Quermassintegrale for hyperbolic convex domains. It is easy to see that the average of the projections onto geodesic subspaces by some origin, depends on the choice of this origin. However, one can take (4) as a definition. For a (geodesically) convex domain $Q \subset \mathbb{H}^n$ we define

$$W_r(Q) := \frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r} \chi(L\cap Q)\,\mathrm{d}L,$$

where $\mathcal{L}_r$ is the space of r-dimensional totally geodesic subspaces $L \subset \mathbb{H}^n$, and $\mathrm{d}L$ is the natural (invariant) measure on it (cf. [2]). As in the euclidean case we take $W_0(Q) = \mathrm{vol}(Q)$, and $W_n(Q) = O_{n-1}/n$. With these definitions, the Quermassintegrale do not coincide with the total mean curvatures, but they are closely related (cf. [11])

$$M_r(\partial Q) = n\left(W_{r+1}(Q) + \frac{r}{n-r+1}W_{r-1}(Q)\right). \tag{5}$$

Therefore we are concerned with inequalities between Quermassintegrale, and also between total mean curvatures. The main results are the following.

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