Available online at www.sciencedirect.com

SCIENCE DIRECT

Differential Geometry and its Applications 22 (2005) 355–378

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

www.elsevier.com/locate/difgeo

# Geometric connections and geometric Dirac operators on contact manifolds ☆

Liviu I. Nicolaescu

*Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA*

Received 18 July 2003; received in revised form 3 June 2004

Communicated by R.L. Bryant

**Abstract**

We construct some natural metric connections on metric contact manifolds compatible with the contact structure and characterized by the Dirac operators they determine. In the case of CR manifolds these are invariants of a fixed pseudo-hermitian structure, and one of them coincides with the Tanaka–Webster connection.


*MSC:* 53B05; 53C15; 53D10; 53D15

*Keywords:* Contact manifolds; CR manifolds; Almost hermitian manifolds; Laplacians; Dirac operators; Metric connections

## Introduction

This work has its origin in our attempt to better understand the nature of Seiberg–Witten monopoles on contact 3-manifolds. The main character of this story is a metric contact manifold $(M, g, \eta, J)$, where $g$ is a Riemann metric, and $\eta$ is a contact form and $J$ is an almost complex structure on $V := \ker \eta$ such that

$$g(X, Y) = d\eta(X, JY), \quad \forall X, Y \in C^\infty(V).$$

☆ This work was partially supported by NSF grant DMS-0303601.
*E-mail address:* nicolaescu.1@nd.edu.

0926-2245/$ – see front matter 
doi:10.1016/j.difgeo.2005.03.008

We are interested in the local differential geometry of such a manifold and in particular, we seek "natural" connection in the tangent bundle $TM$.

Gauge theory suggests that a "natural" connection ought to be compatible with $g$ and $J$. We will refer to these as *metric contact connections*. These requirements alone still leave open a wide range of choices. On the other hand such manifolds are also equipped with some natural elliptic partial differential operators. For the simplicity of the exposition assume $M$ is equipped with a *spin* structure with associated complex spinor bundle $\mathbb{S}$. Every metric connection $\nabla$ on $M$ induces a Dirac type operator

$$\mathfrak{D}(\nabla): C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S}).$$

A metric connection $\nabla$ is called *balanced* if $\mathfrak{D}(\nabla)$ is symmetric. Two connections $\nabla^i$, $i = 0, 1$, will be called *Dirac equivalent* if $\mathfrak{D}(\nabla^0) = \mathfrak{D}(\nabla^1)$. The first question we address in this paper is the existence of a metric contact connection Dirac equivalent with the Levi-Civita connection.

On the other hand, a metric contact manifold is equipped with a natural elliptic, first order operator $\mathcal{H}$ resembling very much the Hodge–Dolbeault operator on a complex manifold (see Section 3.3 for more details). This operator acts on the sections of the complex spinor bundle $\mathbb{S}_c$ associated to the canonical *spin*$^c$ structure determined by the contact structure. A metric contact connection $\nabla$ induces a (geometric) Dirac operator $\mathfrak{D}_c(\nabla)$ on $C^\infty(\mathbb{S}_c)$.

The second question we address in this paper concerns the existence of a metric contact connection $\nabla$ such that $\mathfrak{D}_c(\nabla) = \mathcal{H}$. We say that such a connection is *adapted to* $\mathcal{H}$.

To address these questions we rely on the work P. Gauduchon (see [4] or Section 2.1), concerning hermitian connections on almost-hermitian manifolds. More precisely, to implement Gauduchon's results we will regard $M$ as boundary of certain (possible non-complete) almost hermitian manifolds. We will concentrate only on two cases frequently arising in gauge theory.

- The symplectization $\tilde{M} = \mathbb{R}_+ \times M$ with symplectic form $\omega = \hat{d}(t\eta)$, metric $\tilde{g} = dt^2 + \eta^{\otimes 2} + tg|_V$, and almost complex structure $\tilde{J}$.
- The cylinder $\hat{M} = \mathbb{R} \times M$ with metric $\hat{g} = dt^2 + g$ and almost complex structure $\hat{J}$ defined by $\hat{J}\partial_t = \xi$, $\hat{J}|_V = J$.

To answer the second question we use the cylinder case and a certain natural perturbation of the first canonical connection on $(T\hat{M}, \hat{g}, \hat{J})$. This new connection on $T\hat{M}$ preserves the splitting $T\hat{M} = \mathbb{R}\partial_t \oplus TM$ and induces a connection on $TM$ with the required properties (see Section 3.1). Moreover, when $M$ is a CR manifold this connection coincides with the Tanaka–Webster connection, [10,13].

To answer the first question we use the symplectization $\tilde{M}$ and a natural perturbation of the Chern connection on $T\tilde{M}$. We obtain a new connection on $\tilde{M}$ whose restriction to $\{1\} \times M$ is a contact connection (see Section 3.4). When $M$ is CR this contact connection is also CR, but it never coincides with the Tanaka–Webster connection. We are not aware whether this contact connection has been studied before.

**Theorem.** (a) *On any metric contact manifold there exists a* balanced contact connection *adapted to* $\mathcal{H}$ *and a* balanced contact connection *Dirac equivalent to the Levi-Civita connection. If the manifold is CR these connections are also CR.*

(b) *On a CR manifold each Dirac equivalence class of balanced connections contains at most one CR connection. Moreover, the Tanaka–Webster connection is the unique balanced CR connection adapted to* $\mathcal{H}$.