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How tightly can you fold a sphere?

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Abstract

Consider a compact, connected Lie group G acting isometrically on a sphere S^n of radius 1. The quotient of S^n by this group action, S^n/G, has a natural metric on it, and so we may ask what are its *diameter* and q-*extents*. These values have been computed for cohomogeneity one actions on spheres. In this paper, we compute the diameters, extents, and several q-extents of cohomogeneity two orbit spaces resulting from such actions, and we also obtain results about the q-extents of Euclidean disks. Additionally, via a simple geometric criterion, we can identify which of these actions give rise to a decomposition of the sphere as a union of disk bundles. In addition, as a service to the reader, we give a complete breakdown of all the isotropy subgroups resulting from cohomogeneity one and two actions.

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1. Introduction

The diameters of spherical Alexandrov spaces (that is, quotients of the unit sphere under actions by compact, connected Lie groups) have been calculated for some special cases. In particular, the diameters of space forms (quotients of the unit sphere by finite groups acting properly discontinuously, freely and isometrically) and those of curvature one orbifolds (quotients of the unit sphere by finite groups acting isometrically, but not necessarily freely and properly discontinuously) have been calculated by McGowan [11] and Greenwald [4], respectively. Greenwald also showed that there is a lower bound for the diameters of spherical Alexandrov spaces, but did not explicitly calculate it [4]. Flach proved the existence of a lower bound for the diameters of δ-pinched manifolds, i.e., those whose sectional curvature K satisfies $\delta \leqslant K \leqslant 1$ [3], whose dimension is sufficiently large.

On the other end of the spectrum, the lengths (hence, diameters) of one-dimensional orbit spaces produced by actions on spheres have been determined (cf. [5,14]). An action whose orbit space S^n/G has dimension k is said to be an action of *cohomogeneity* k. The orbit space of a cohomogeneity one action on a sphere is necessarily an interval, and its length is π/n where n corresponds to the number of principal curvatures of the action (and equals 1, 2, 3, 4 or 6).

In this paper, we calculate the diameters and the extents for cohomogeneity two spherical Alexandrov spaces as what we consider the next most natural step in terms of what has already been accomplished. In addition, we calculate several q-extents for the various spaces and provide a method to find all the other q-extents. Incidentally to accomplishing this goal, we also calculate all the isotropy subgroups (not merely the principal ones) for all cohomogeneity one and two actions on spheres. The diameters and q-extents of spherical Alexandrov spaces are of special interest: in the study of transformation groups, it is often of great importance to know the diameter of the space of directions to a given point of isotropy in the manifold M^n under consideration. This information has been to used to great advantage in [9] and [7] in order to determine the total number of singular points in a given manifold of positive sectional curvature admitting an isometric action by a compact, connected Lie group G. From this information alone, it is often possible to determine the topological structure of the manifold as in the following equivariant sphere theorem:

Equivariant sphere theorem [6,7]. *Let M be a closed manifold with $\sec(M) > 0$ on which G acts (almost) effectively by isometries. Suppose $p_0, p_1 \in M$ are points such that $\operatorname{diam} S_{\bar{p}_i} \leqslant \pi/4$, $i = 0, 1$, where $S_{\bar{p}_i}$ is the space of directions at $\bar{p}_i$ in M/G. Then M can be exhibited as*

$$M = D\big(G(p_0)\big) \cup_E D\big(G(p_1)\big),$$

where $D(G(p_i))$, $i = 0, 1$, are tubular neighborhoods of the p_i-orbits and $E = \partial D(G(p_0)) = \partial D(G(p_1))$. In particular, M is homeomorphic to the sphere if $G(p_i) = p_i$, i.e., if p_i, $i = 1, 0$, are isolated fixed points of G and $\operatorname{diam} S_{\bar{p}_i} \leqslant \pi/4$.

Of course, as we will see, a diameter of $\pi/4$ or less in the known cases occurs with great frequency although not as frequently as $\pi/2$ or π.

Moreover, for spherical space forms, it was shown that a lower limit for the diameter is given by $\frac{1}{2}\arccos(\frac{1}{\sqrt{3}}\tan(\frac{3\pi}{10}))$. This lower bound is optimal and is achieved in dimension three.