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Differential Geometry and its Applications 23 (2005) 114–127

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

www.elsevier.com/locate/difgeo

4-dimensional almost Kähler manifolds and L^2-scalar curvature functional

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Received 28 April 2004

Available online 29 June 2005

Communicated by K. Fukaya

Abstract

In this paper, we consider the integrability of compact almost Kähler 4-manifolds for which a certain part of the divergence of the Weyl conformal tensor vanishes. Proof of our main theorem gives another one of the theorem of M. Itoh [Intern. J. Math. 15 (2004) 573–580] which gives partial answer to the Goldberg conjecture in case of 4-dimensional negative scalar curvature.

MSC: 53C25; 53C55

Keywords: Almost Hermitian manifold; Almost Kähler manifold; Kähler manifold; Weyl conformal tensor

1. Introduction

Let (M^{2n}, J, g) be an almost Hermitian manifold with an almost complex structure J and a J-invariant metric g on M. (M, J, g) is an *almost Kähler* manifold if the Kähler form ω, defined by $\omega(X, Y) = g(JX, Y)$, is a closed 2-form. On the other hand, (M, J, g) is a *Kähler* manifold if J is integrable and ω is closed, or equivalently if J is parallel with respect to the Levi-Civita connection ∇ of g. Kähler manifolds are almost Kähler manifolds. However, in general the converse is not true. Some examples of almost Kähler manifolds whose almost complex structure is not integrable were obtained.

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2005.05.004

S.I. Goldberg [4] considered the integrability of an almost Kähler structure in terms of some curvature conditions and stated a strong conjecture; *The almost complex structure of a compact almost Kähler Einstein manifold is integrable*.

The above conjecture is partially solved by K. Sekigawa [8,9] in case of non-negative scalar curvature. Some advanced results about integrability theorem of almost Kähler manifolds with some curvature conditions are obtained. However, the conjecture is still open in the case of negative scalar curvature.

In this article, we consider the integrability of almost Kähler structure on a Riemannian manifold satisfying the condition $\pi(\delta W) = 0$ which implies that a certain part of the divergence of the Weyl conformal tensor vanishes.

The Weyl conformal tensor W of a Riemannian m-manifold (M^m, g) is a part of the Riemannian curvature tensor R of g defined by

$$R = W + \frac{1}{m-2} Ric_0 \owedge g + \frac{s}{2m(m-1)} g \owedge g \tag{1}$$

where $s = s_g$ is the scalar curvature and $Ric_0 = Ric - \frac{s}{m} g$ is the trace-free Ricci tensor. Here the symbol $\owedge$ is the Nomizu–Kulkarni product of symmetric $(0,2)$-tensors generating a curvature type tensor.

The divergence of W, denoted by δW, is a tensor field of type $(0,3)$ defined by

$$\delta W(X_1, X_2, X_3) = -\operatorname{Trace}_g\big\{(X, Y) \to (\nabla_X W)(Y, X_1, X_2, X_3)\big\}$$

for vector fields X_1, X_2, X_3. By the second Bianchi identity, the divergence δW of the Weyl curvature tensor can be expressed as

$$\delta W = -\frac{m-3}{m-2} d_R\left(Ric - \frac{s}{2(m-1)} g\right) \tag{2}$$

where $d_R : \Gamma(M; \Lambda^1 M \otimes \Lambda^1 M) \to \Gamma(M; \Lambda^1 M \otimes \Lambda^2 M)$ is defined by

$$d_R T(X, Y, Z) = (\nabla_Y T)(X, Z) - (\nabla_Z T)(X, Y) \tag{3}$$

for $T \in \Gamma(M; \Lambda^1 M \otimes \Lambda^1 M)$. Therefore, any Riemannian manifold with parallel Ricci tensor satisfies $\delta W = 0$; in particular Einstein manifolds and locally symmetric spaces do. Of course, any conformally flat manifold satisfies this condition (see [2, p. 440]).

Since the divergence δW of the Weyl conformal tensor can be regarded as a section of $\Lambda^1 M \otimes \Lambda^2 M$, according to the decomposition of $\Lambda^2 M$, δW also decomposes into several parts.

If there exists an almost complex structure J on (M, g) and g is J-invariant, then the usual type decomposition $\Lambda^2 M \otimes \mathbb{C} = \Lambda^{2,0} M \oplus \Lambda^{1,1} M \oplus \Lambda^{0,2} M$ of complexified 2-forms induces the decomposition

$$\Lambda^2 M = \mathbb{R}\omega \oplus (\Lambda_0^{1,1} M)_{\mathbb{R}} \oplus (\Lambda^{2,0} M \oplus \Lambda^{0,2} M)_{\mathbb{R}} \tag{4}$$

where $\mathbb{R}\omega$ is the line bundle generated by the Kähler form ω and $\Lambda_0^{1,1} M$ is the orthogonal complement of $\mathbb{R}\omega$ in $\Lambda^{1,1} M$. We denote $(\Lambda^{2,0} M \oplus \Lambda^{0,2} M)_{\mathbb{R}}$ by LM and the projection from $\Lambda^2 M$ to LM by π;

$$LM := (\Lambda^{2,0} M \oplus \Lambda^{0,2} M)_{\mathbb{R}}, \qquad \pi : \Lambda^2 M \longrightarrow LM.$$

In dimension 4, as an endomorphism of $\Lambda^2 M$ the Weyl conformal tensor W commutes with the Hodge star operator $*$, i.e., W preserves the decomposition $\Lambda^2 M = \Lambda_+^2 M \oplus \Lambda_-^2 M$ where $\Lambda_+^2 M$ and $\Lambda_-^2 M$ are the eigenspaces corresponding to the eigenvalues $+1$ and -1 of the operator $*$, respectively. We denote the restriction of W to $\Lambda_+^2 M$ (or $\Lambda_-^2 M$) by W^+ (or W^-, respectively) called the (anti-)self-dual Weyl