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Vector bundles over Grassmannians and the skew-symmetric curvature operator

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Abstract

A pseudo-Riemannian manifold is said to be spacelike Jordan IP if the Jordan normal form of the skew-symmetric curvature operator depends upon the point of the manifold, but not upon the particular spacelike 2-plane in the tangent bundle at that point. We use methods of algebraic topology to classify connected spacelike Jordan IP pseudo-Riemannian manifolds of signature (p,q), where $q \geqslant 11$, $p \leqslant \frac{q-6}{4}$ and where the set $\{q,\dots,q+p\}$ does not contain a power of 2.

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0. Introduction

Let M be a manifold of signature (p,q). Let $T_P M$ denote the tangent space to M at a point P and let the metric on M be denoted by $(\cdot,\cdot)$. Furthermore, let R be the Riemann curvature tensor of M with respect to the Levi-Civita connection ∇. To every non-degenerate oriented 2-dimensional plane π in

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$T_P M$ we can associate *the skew-symmetric curvature operator*:

$$R(\pi) := \left|(x,x)(y,y) - (x,y)^2\right|^{-1/2} R(x,y),$$

where $\{x, y\}$ is any of the oriented bases for π. It can be shown (see Lemma 1.9.1 in [3]) that the operator $R(\pi)$ is well-defined, i.e., that the definition given above is independent of the chosen oriented basis for π. In differential geometry one studies the eigenvalue structure of the family of operators

$$\left\{R(\pi) : T_P M \to T_P M \mid \pi \text{ oriented non-degenerate 2-plane in } T_P M\right\}.$$

One typically imposes one of the following conditions:

(1) The eigenvalue structure of $R(\pi)$ does not depend on the choice of spacelike plane $\pi \subset T_P M$. In this case we say that R is *spacelike IP at P*. If R is spacelike IP at every point $P \in M$, we say that M is a *spacelike IP manifold*.
(2) The eigenvalue structure of $R(\pi)$ does not depend on the choice of timelike plane π. In this case we talk about *timelike IP manifolds*. Similarly, we talk about *mixed IP*—the case where the eigenvalue structure of $R(\pi)$ does not depend on the choice of plane π of signature $(1, 1)$.
(3) *Jordan* (*spacelike, timelike or mixed*) *IP*: manifolds where not only the eigenvalue structure does not change with π, but the Jordan normal form as well.

Four-dimensional Riemannian Jordan IP manifolds were first studied and classified by Ivanov and Petrova in [7]. Subsequently, Gilkey, Leahy and Sadofsky [4] classified Riemannian (Jordan) IP manifolds whose dimension is at least five and not equal to seven or eight. They showed that the only examples of Riemannian Jordan IP manifolds are manifolds of constant sectional curvature and certain warped products of an open interval by a manifold of constant sectional curvature.

The two examples generalize to the higher signature setting as follows:

Example 0.1.

(1) Let V be a vector space and let $g(\cdot,\cdot)$ be a non-degenerate innerproduct on V. Let $K \in \mathbb{R}$ with $K \neq 0$. Equip the pseudo-sphere

$$S := \left\{v \in V \mid g(v, v) = 1/K\right\}$$

with the metric induced by the inclusion $S \subset V$. Manifold S is a spacelike Jordan IP manifold; for details the reader is referred to [3].

(2) Let S be the pseudo-sphere discussed in example (1); for $K = 0$ let S be the flat space. Let $\varepsilon \in \{\pm 1\}$ and let $A, B \in \mathbb{R}$ satisfy the condition $A^2 - 4\varepsilon K B \neq 0$. Set $f(t) := \varepsilon K t^2 + At + B$ and choose a connected open interval $I \subset \mathbb{R}$ so that $f(t) \neq 0$ on I. We consider

$$M := I \times S \quad \text{with } ds_M^2 := \varepsilon\, dt^2 + f(t)\, ds_S^2.$$

Manifold M is a spacelike Jordan IP manifold; the reader is referred to [3] for further details.

The classification result of Gilkey, Leahy and Sadofsky [4] has been extended to the higher signature setting (see Zhang [12,13], and Gilkey and Zhang [5]) as follows: