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ParaHermitian and paraquaternionic manifolds

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Abstract

A set of canonical paraHermitian connections on an almost paraHermitian manifold is defined. ParaHermitian version of the Apostolov–Gauduchon generalization of the Goldberg–Sachs theorem in General Relativity is given. It is proved that the Nijenhuis tensor of a Nearly paraKähler manifolds is parallel with respect to the canonical connection. Salamon's twistor construction on quaternionic manifold is adapted to the paraquaternionic case. A hyper-paracomplex structure is constructed on Kodaira–Thurston (properly elliptic) surfaces as well as on the Inoe surfaces modeled on Sol_1^4. A locally conformally flat hyper-paraKähler (hypersymplectic) structure with parallel Lee form on Kodaira–Thurston surfaces is obtained. Anti-self-dual non-Weyl flat neutral metric on Inoe surfaces modeled on Sol_1^4 is presented. An example of anti-self-dual neutral metric which is not locally conformally hyper-paraKähler is constructed.

MSC: 53C15; 5350; 53C25; 53C26; 53B30

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1. Introduction

We study the geometry of structures on a differentiable manifold related to the algebra of paracomplex numbers as well as to the algebra of paraquaternions together with a naturally associated metric which is necessarily of neutral signature. These structure lead to the notion of almost paraHermitian manifold, in even dimension, as well as to the notion of almost paraquaternionic and hyper-paraHermitian manifolds in dimensions divisible by four. Some of these spaces, hyper-paracomplex and hyper-paraHermitian manifolds, become attractive in theoretical physics since they play a role in string theory [11,40,41,44,55] and integrable systems [26].

Almost paraHermitian geometry is a topic with many analogies with the almost Hermitian geometry and also with differences. In the present note we show that a lot of local and some of the global results in almost Hermitian manifolds carry over, in the appropriately defined form, to the case of almost paraHermitian spaces.

We define a set of canonical paraHermitian connections on an almost paraHermitian manifold and use them to describe properties of 4-dimensional paraHermitian and 6-dimensional Nearly paraKähler spaces.

We present a paraHermitian analogue of the Apostolov–Gauduchon generalization [6] of the Goldberg–Sachs theorem in General Relativity (see e.g. [57]) which relates the Einstein condition to the structure of the positive Weyl tensor in dimension 4. Namely, we prove

Theorem 1.1. *Let (M, g, P) be a 4-dimensional paraHermitian manifold. Let W^+ be the self-dual part of the Weyl tensor and θ be the Lee 1-form. The following conditions are equivalent*:

(a) *The 2-form $d\theta$ is anti-self-dual, $d\theta^+ = 0$;*
(b) *$W_2^+ = 0$, equivalently, the fundamental 2-form is an eigen-form of W^+;*
(c) *$(\delta W^+)^- = 0$, equivalently, $(\delta W)(X^{1,0}; Y^{1,0}, Z^{1,0}) = 0$.*

Corollary 1.2. *Assume that the Ricci tensor ρ of a paraHermitian 4-manifold is P-anti-invariant, $\rho(PX, PY) = -\rho(X, Y)$. Then $d\theta$ is anti-self-dual 2-form, $d\theta^+ = 0$.*

In particular, on a paraHermitian Einstein 4-manifold the fundamental 2-form is an eigen-form of the positive Weyl tensor.

It turns out that any conformal class of neutral metrics on an oriented 4-manifold is equivalent to the existence of a local almost hyper-paracomplex structure, i.e., a collection of anti-commuting almost complex structure and almost para-complex structure. Using the properties of the Bismut connection, we derive that the integrability of the almost hyper-paracomplex structure leads to the anti-self-duality of the corresponding conformal class of neutral metrics (Theorem 6.2). Applying this result to invariant hyper-paracomplex structure on 4-dimensional Lie groups [4,22] we find explicit anti-self-dual non-Weyl flat neutral metrics on some compact 4-manifolds. Some of these metrics seem to be new.

We apply our considerations to Kodaira–Thurston complex surfaces modeled on $S^1 \times \widetilde{SL(2, \mathbb{R})}$ (properly elliptic surfaces) as well as to the Inoe surfaces modeled on Sol_1^4 in the sense of [65]. These surfaces do not admit any (para) Kähler structure [18,58,65]. It is also known that these surfaces do not support a hyper-complex structure [23,49].

In contrast, we obtain