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Reduction of Jacobi manifolds via Dirac structures theory

Fani Petalidou [a], Joana M. Nunes da Costa [b,*]

[a] *Faculty of Sciences and Technology, University of Peloponnese, 22100 Tripoli, Greece*
[b] *Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal*

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Abstract

We first recall some basic definitions and facts about Jacobi manifolds, generalized Lie bialgebroids, generalized Courant algebroids and Dirac structures. We establish an one–one correspondence between reducible Dirac structures of the generalized Lie bialgebroid of a Jacobi manifold (M, Λ, E) for which 1 is an admissible function and Jacobi quotient manifolds of M. We study Jacobi reductions from the point of view of Dirac structures theory and we present some examples and applications.

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1. Introduction

The concept of a *Dirac structure* on a differentiable manifold M was introduced by T. Courant and A. Weinstein in [2] and developed by T. Courant in [3]. Its principal aim is to present a unified framework for the study of pre-symplectic forms, Poisson structures and foliations. More specifically, a *Dirac structure* on M is a subbundle $L \subset TM \oplus T^*M$ that is maximally isotropic with respect to the canonical symmetric bilinear form on $TM \oplus T^*M$ and satisfies a certain integrability condition. In order to for-

* Corresponding author.
E-mail addresses: petalido@uop.gr (F. Petalidou), jmcosta@mat.uc.pt (J.M. Nunes da Costa).

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mulate this integrability condition, T. Courant defines a bilinear, skew-symmetric, bracket operation on the space $\Gamma(TM \oplus T^*M)$ of smooth sections of $TM \oplus T^*M$ which does not satisfy the Jacobi identity. The nature of this bracket was clarified by Z.-J. Liu, A. Weinstein and P. Xu in [18] by introducing the structure of a *Courant algebroid* on a vector bundle E over M and by extending the notion of a Dirac structure to the subbundles $L \subset E$. The most important example of Courant algebroid is the direct sum $A \oplus A^*$ of a Lie bialgebroid (A, A^*) over a smooth manifold M [22].

Alan Weinstein and his collaborators have studied several problems of Poisson geometry via Dirac structures theory. In [19], Z.-J. Liu et al. establish an one–one correspondence between Dirac subbundles of the double $TM \oplus T^*M$ of the triangular Lie bialgebroid (TM, T^*M, Λ) defined by a Poisson structure Λ on M and Poisson structures on quotient manifolds of M. Using this correspondence and the results concerning the pull-backs Dirac structures under Lie algebroid morphisms, Z.-J. Liu constructs in [20] the Poisson reduction in terms of Dirac structures.

On the other hand, it is well known that the notion of *Jacobi manifold*, i.e., a differentiable manifold M endowed with a bivector field Λ and a vector field E satisfying an integrability condition, introduced by A. Lichnerowicz in [17], is a rich geometrical notion that generalizes the Poisson, symplectic, contact and co-symplectic manifolds. Thus, it is natural to research a simple interpretation of Jacobi manifolds by means of Dirac structures. A first approach of this problem is presented in [32] by A. Wade. Taking into account that to any Jacobi structure (Λ, E) on M is canonically associated a generalized Lie bialgebroid structure on $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ [10], she considers the Whitney sum $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, introduces the notion of $\mathcal{E}^1(M)$*-Dirac structures* by extending the Courant's bracket to the space $\Gamma(\mathcal{E}^1(M))$ of smooth sections of $\mathcal{E}^1(M)$ and shows that the graph of the vector bundle morphism $(\Lambda, E)^\#: T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ is a Dirac subbundle of $\mathcal{E}^1(M)$. But the extended bracket does not endow $\mathcal{E}^1(M)$ with a Courant algebroid structure. A second approach of the problem is the one proposed by the second author and J. Clemente-Gallardo in the recent paper [30]. They introduce the notions of *generalized Courant algebroid* (which is equivalent to the notion of *Courant–Jacobi algebroid* independently defined by J. Grabowski and G. Marmo in [8]) and of *Dirac structure for a generalized Courant algebroid* and give several connections between Dirac structures for generalized Courant algebroids and Jacobi manifolds. We note that the construction of [30] includes as particular case the one of Wade and that the main example of generalized Courant algebroid over M is the direct sum of a generalized Lie bialgebroid over M.

In the present work, by using the results mentioned above, we establish a reduction theorem of Jacobi manifolds (Theorem 5.2). It is well known that there are already several geometric and algebraic treatments of the Jacobi reduction problem (see, for instance, [9,26–28]). But, it is an original goal of the Dirac structures theory to describe Jacobi reduction and to construct a more general framework for the study of the related problems concerning the projection of Jacobi structures and the existence of Jacobi structures on certain submanifolds of Jacobi manifolds. Precisely, on the way to our principal result, we construct an one to one correspondence between Dirac subbundles, satisfying a certain regularity condition, of the double $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, where M is a Jacobi manifold, and quotient Jacobi manifolds of M (Theorem 4.14). Also, the reduction theorem (Theorem 5.2) allows us to state sufficient conditions under which a submanifold N of (M, Λ, E) inherits a Jacobi structure, that include as particular cases the results presented in [4,12].

The paper is organized as follows. In Sections 2 and 3 we recall some basic definitions and results concerning, respectively, Jacobi structures, generalized Lie bialgebroids and Dirac structures for generalized Courant algebroids. In Section 4 we establish a correspondence between Dirac structures and