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Differential Geometry and its Applications 23 (2005) 305–326

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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Riemannian geometry of finite rank positive operators

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Received 11 August 2003; received in revised form 20 September 2004

Available online 9 August 2005

Communicated by P. Michor

Abstract

A riemannian metric is introduced in the infinite dimensional manifold Σ_n of positive operators with rank $n < \infty$ on a Hilbert space H. The geometry of this manifold is studied and related to the geometry of the submanifolds Σ_p of positive operators with range equal to the range of a projection p (rank of $p = n$), and $\mathcal{P}_p$ of selfadjoint projections in the connected component of p. It is shown that these spaces are complete in the geodesic distance.

MSC: 58B20; 57N20; 58B05

Keywords: Positive operator; Finite rank projection

1. Introduction

The space $M_n^+(\mathbb{C})$ of positive definite (invertible) matrices is a differentiable manifold, in fact an open subset of the real euclidean space of hermitian matrices. Let x, y be hermitian matrices and a positive definite, the formula

$$\langle x, y\rangle_a = \operatorname{tr}(xa^{-1}ya^{-1})$$

endows $M_n^+(\mathbb{C})$ with a riemannian metric, which makes it a negatively curved, complete metric space. This fact is well known and has been used in a variety of contexts. For example, in interpolation theory

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2005.06.004

of Banach and Hilbert spaces [9,19], in partial differential equations [18], or in mathematical physics [10,16,20]. It has also been generalized to infinite dimensions, i.e., Hilbert spaces and operator algebras: [4,7,8,20].

The purpose of this paper is to introduce a riemannian structure in the set Σ_n of positive operators of finite (fixed) rank n on an infinite dimensional Hilbert space H. Note that even though $n < \infty$, this set Σ_n is infinite dimensional. Corach et al. [4,5] considered a Finsler structure for positive non-invertible operators with fixed *range*. We go one step further fixing only the *rank*. The condition that the rank is fixed ensures that for all $a \in \Sigma_n$, the projections onto their ranges, which we denote by $\rho(a)$, are unitarily equivalent.

In particular, if p is a projection with rank n, then the connected component $\mathcal{P}_p$ of p in the space of projections, lies inside Σ_n. Also inside Σ_n lies Σ_p, the space of positive operators with range equal to the range of p. Apparently, Σ_p identifies with $M_n^+(\mathbb{C})$. We shall introduce a riemannian metric in Σ_n, which naturally generalizes the metric given above for $M_n^+(\mathbb{C})$, and which restricted to Σ_p makes the identification of this space with $M_n^+(\mathbb{C})$ isometric. Moreover, when restricted to $\mathcal{P}_p$, one obtains the trace inner product of this space. Our main result on Σ_n Theorem 6.9 states that, though we lose the negative curvature properties for positive operators (because $\mathcal{P}_p$ inside Σ_n is positively curved), Σ_n is a complete metric space for the geodesic distance.

Let us fix some notation. Let H be a Hilbert space, $\mathcal{U}(H)$ and $\mathcal{G}l(H)$ the Banach–Lie groups of, respectively, unitary and invertible operators of H. Throughout this paper $\|x\|$ will denote the usual operator norm of $x \in \mathcal{B}(H)$. Fix $n < \infty$, and let p a projection with rank n, and consider the following sets:

- Σ_n the set of positive operators with rank n.
- Σ_p the set of positive operators with range equal to the range of p.
- $\mathcal{I}_p$ the set of partial isometries with initial space equal to the range of p.

Clearly these sets $\Sigma_p \subset \Sigma_n$ and $\mathcal{I}_p$ are subsets of $\mathcal{B}_2(H)$, the class of Hilbert–Schmidt operators of H. Denote by $\mathcal{P}$ the set of projections acting on H, and by $\mathcal{P}_p$ the connected component (in the norm topology) of p in $\mathcal{P}$, which coincides with the unitary orbit of p, $\{upu^*\colon u \text{ unitary in } H\}$. The three sets of the above list and $\mathcal{P}_p$ will be considered with the inner product topology of $\mathcal{B}_2(H)$. Since these are sets of finite rank operators, this topology coincides there with the operator norm topology of $\mathcal{B}(H)$.

A relevant feature in this study is the map

$$\rho : \Sigma_n \to \mathcal{P}_p,$$

$\rho(a) =$ projection onto the range of a. This map is continuous due to the fact that $n < \infty$. Moreover, it was shown in [5] that it is differentiable. In this paper we revise the differentiable structure of Σ_n, $\mathcal{P}_p$ and $\mathcal{I}_p$. We introduce a riemannian metric in Σ_n, based on the trace of $\mathcal{B}_2(H)$, and consider geometric problems therein. When restricted to the submanifold Σ_p of positive operators with fixed range $p(H)$, one obtains the well studied non-positive curvature connection for the set of positive invertible operators [7].

The contents of the paper are as follows. In Section 2 we revise the riemannian geometry of $\mathcal{P}_p$. As it turns out, the connection looks formally identical to the reductive connection for the space of projections in an abstract C*-algebra [6,14,15]. Then one can profit from the computations done there: geodesics, curvature tensor, etc. Here we establish that $\mathcal{P}_p$ is complete. In Section 3 we consider $\mathcal{I}_p$ with the metric