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A strong convergence theorem for relatively nonexpansive mappings in a Banach space

Shin-ya Matsushita, Wataru Takahashi*

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo, 152-8552, Japan

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Abstract

In this paper, we prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.

MSC: primary 47H09; secondary 47H05

Keywords: Relatively nonexpansive mapping; Nonexpansive mapping; Asymptotic fixed point; Generalized projection; Maximal monotone operator

1. Introduction

Let E be a smooth Banach space and let E^* be the dual of E. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$

* Corresponding author. Fax: +81 3 5734 3208.
E-mail addresses: shinya2@is.titech.ac.jp (S. Matsushita), wataru@is.titech.ac.jp (W. Takahashi).

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the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [13] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [3–5] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [3–5]. On the other hand, Nakajo and Takahashi [9] obtained strong convergence theorems for nonexpansive mappings in a Hilbert space. In particular, they studied the strong convergence of the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S x_n, \\ C_n = \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \; n = 0, 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, S is a nonexpansive mapping from C into itself and $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$.

Motivated by Nakajo and Takahashi [9], our purpose in this paper is to prove a strong convergence theorem for relatively nonexpansive mappings in a Banach space. Using this result, we also discuss the problem of strong convergence concerning nonexpansive mappings in a Hilbert space and maximal monotone operators in a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [6,12,16,17]. A Banach space E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is