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Interlacing property for B-splines

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Abstract

We prove that the zeros of the derivatives of any order of a B-spline are increasing functions of its interior knots. We then prove that if the interior knots of two B-splines interlace, then the zeros of their derivatives of any order also interlace. The same results are obtained for Chebyshevian B-splines.

Keywords: Markov interlacing property; Extended complete Chebyshev spaces; B-splines; Divided differences

1. Introduction

In 1892, Vladimir Markov established the following lemma, now known as the Markov interlacing property.

Lemma 1 (*Markov [7]*). *If the zeros of the polynomial* $p := (\bullet - t_1) \cdots (\bullet - t_n)$ *and the zeros of the polynomial* $q := (\bullet - s_1) \cdots (\bullet - s_n)$ *interlace, that is*

$$t_1 \leqslant s_1 \leqslant t_2 \leqslant s_2 \leqslant \cdots \leqslant t_{n-1} \leqslant s_{n-1} \leqslant t_n \leqslant s_n,$$

then the zeros $\tau_1 \leqslant \cdots \leqslant \tau_{n-1}$ *of* p' *and the zeros* $\sigma_1 \leqslant \cdots \leqslant \sigma_{n-1}$ *of* q' *also interlace, that is*

$$\tau_1 \leqslant \sigma_1 \leqslant \tau_2 \leqslant \sigma_2 \leqslant \cdots \leqslant \tau_{n-1} \leqslant \sigma_{n-1}.$$

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Moreover, if $t_1 < \cdots < t_n$ *and if* $t_i < s_i$ *at least once, then the zeros of* p' *and the zeros of* q' *strictly interlace, that is*

$$\tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \cdots < \tau_{n-1} < \sigma_{n-1}.$$

This lemma plays a major role in the original proof of the Markov inequality [7] and in some of its simplifications, e.g. [2,12]. The interlacing property for perfect splines [1], likewise, is essential in the proof of Markov-type inequalities for oscillating perfect splines [4].

Bojanov remarked that the Markov interlacing property for polynomials is equivalent to a certain monotonicity property, namely

Each zero of the derivative of a polynomial $p := (\bullet - x_1)\cdots(\bullet - x_n)$ *is a strictly increasing function of any* x_j *on the domain* $x_1 < \cdots < x_n$.

He proved [1] this equivalence even for generalized polynomials with respect to a Chebyshev system (satisfying certain conditions), and then obtained the Markov interlacing property for generalized polynomials by showing the monotonicity property.

Bojanov's arguments were somehow similar to the ones used by Vidensky when he gave, in 1951, the following general lemma.

Lemma 2 (*Videnskii [13]*). *Let f and g be two continuously differentiable functions such that any non-trivial linear combination of f and g has at most n zeros counting multiplicity. If the zeros* $t_1 < \cdots < t_n$ *of f and the zeros* $s_1 < \cdots < s_n$ *of g interlace, then* $n-1$ *zeros of* f' *and* $n-1$ *zeros of* g' *strictly interlace.*

In this paper, we aim at proving an interlacing property for B-splines. More precisely, we show that if the interior knots of two polynomial B-splines interlace, then the zeros of their derivatives (of any order) also interlace. In Section 2, we show how this can be derived from what we call the monotonicity property, namely

Each zero of $N^{(l)}_{t_0,\ldots,t_{k+1}}$, $1 \leqslant l \leqslant k-1$, *is a strictly increasing function of any interior knot* t_j, $1 \leqslant j \leqslant k$, *on the domain* $t_0 < t_1 < \cdots < t_k < t_{k+1}$.

This property is proved in Section 3. Next, we generalize these statements to Chebyshevian B-splines. To this end, we need various results which are scattered around the literature and are recalled in Sections 4, 6 and 7. Finally, the proof of the monotonicity property for Chebyshevian B-splines is presented in Section 8.

Our interest in this problem arose from a conjecture regarding the B-spline basis condition number formulated by Scherer and Shadrin [11]. For $\underline{t} = (t_0 < t_1 < \cdots < t_k < t_{k+1})$, with $\omega_{\underline{t}}$ representing the monic polynomial of degree k which vanishes at $t_1, \ldots, t_k$, they asked if it was possible to find a function $\Omega_{\underline{t}}$ vanishing k-fold at t_0 and t_{k+1} and such that the sign pattern of $\Omega^{(l)}_{\underline{t}}$ is the same as the sign pattern of $(-1)^l \omega^{(k-l)}_{\underline{t}}$, $0 \leqslant l \leqslant k$. The hope to choose $\Omega_{\underline{t}}$ as a Chebyshevian B-spline with knots $t_0, \ldots, t_{k+1}$ raised the problem of the monotonicity property. Indeed, the zeros of $\Omega^{(l)}_{\underline{t}}$ should coincide with the zeros of $\omega^{(k-l)}_{\underline{t}}$ and thus should increase with any t_j, $1 \leqslant j \leqslant k$.