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The rate of convergence of q-Bernstein polynomials for $0 < q < 1$ ☆

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Abstract

In the note, we obtain the estimates for the rate of convergence for a sequence of q-Bernstein polynomials $\{B_{n,q}(f)\}$ for $0 < q < 1$ by the modulus of continuity of f, and the estimates are sharp with respect to the order for Lipschitz continuous functions. We also get the exact orders of convergence for a family of functions $f(x) = x^{\alpha}$, $\alpha > 0$, $\alpha \neq 1$, and the orders do not depend on α, unlike the classical case.

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Keywords: q-Bernstein polynomials; Rate of approximation; Modulus of continuity

1. Introduction

Let $q > 0$. For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := \begin{cases} (1-q^k)\big/(1-q), & q \neq 1 \\ k, & q = 1 \end{cases}$$

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and

$$[k]! := \begin{cases} [k]\,[k-1]\cdots[1], & k \geqslant 1 \\ 1, & k = 0 \end{cases}$$

respectively. For the integers $n,\ k,\ n \geqslant k \geqslant 0$, the q-binomial coefficients are defined by (see [3, p. 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

In 1997, Phillips proposed the following q-Bernstein polynomials $B_{n,q}(f,x)$. For each positive integer n, and $f \in C[0,1]$, we define

$$B_{n,q}(f,x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x), \tag{1.1}$$

where it is agreed that an empty product denotes 1 (see [6]). When $q = 1$, $B_{n,q}(f,x)$ reduce to the well-known Bernstein polynomials $B_n(f,x)$:

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

In recent years, the q-Bernstein polynomials have attracted much interest, and a great number of interesting results related to the q-Bernstein polynomials have been obtained (see [2,5–9]). This note is concerned with the quantitative results for the rate of convergence of the q-Bernstein polynomials for $0 < q < 1$. For $f \in C[0,1],\ t > 0$, we define the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\omega(f;t) := \sup_{\substack{|x-y| \leqslant t \\ x,y \in [0,1]}} |f(x) - f(y)|,$$

$$\omega_2(f,t) := \sup_{0 < h \leqslant t} \ \sup_{x \in [0,1-2h]} |f(x+2h) - 2f(x+h) + f(x)|.$$

For fixed $q \in (0,1)$, II'inskii and Ostrovska proved in [2] that for each $f \in C[0,1]$, the sequence $\{B_{n,q}(f,x)\}$ converges to $B_{\infty,q}(f,x)$ as $n \to \infty$ uniformly for $x \in [0,1]$, where

$$B_{\infty,q}(f,x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k) \frac{x^k}{(1-q)^k [k]!} \prod_{s=0}^{\infty} (1 - q^s x), & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases} \tag{1.2}$$

The first author of the note gave the following quantitative result for the rate of convergence of the q-Bernstein polynomials (see [9]):

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| \leqslant c\, \omega_2(f, \sqrt{q^n}\,) \tag{1.3}$$

with $\|\cdot\|$ the uniform norm, here c is an absolute constant. Note that when $f(x) = x^2$, we have (see [9]):

$$\|B_{n,q}(f) - B_{\infty,q}(f)\| = \sup_{x \in [0,1]} \frac{q^n(1-q)}{1-q^n} x(1-x) \asymp q^n \asymp \omega_2(f, \sqrt{q^n}\,),$$