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The finite section method and problems in frame theory

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Abstract

The finite section method is a convenient tool for approximation of the inverse of certain operators using finite-dimensional matrix techniques. In this paper we demonstrate that the method is very useful in frame theory: it leads to an efficient approximation of the inverse frame operator and also solves related computational problems in frame theory. In the case of a frame which is localized w.r.t. an orthonormal basis we are able to estimate the rate of approximation. The results are applied to the reproducing kernel frame appearing in the theory for shift-invariant spaces generated by a Riesz basis.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. A family $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2 \; \forall f \in \mathcal{H}.$$

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Given a frame $\{f_k\}_{k=1}^{\infty}$, the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \; Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k \tag{1}$$

is bounded and invertible, and each $f \in \mathcal{H}$ has the representation

$$f = \sum_{k=1}^{\infty} \langle S^{-1} f, f_k \rangle f_k, \tag{2}$$

see [4,6,15]. In order to use (2) in practice, we need efficient methods to invert the frame operator. The problem of designing finite-dimensional models for approximating the inverse frame operator leads to delicate questions of stability and convergence, cf. [5] and the references cited therein. In this paper we demonstrate that the finite section method, when applied properly, is very useful for this purpose.

We present the general results in Section 2. In Section 3 we apply our findings to two important issues in the theory of shift-invariant spaces generated by a Riesz basis: namely, inversion of the frame operator associated to the reproducing kernel frame, and reconstruction of a function from a set of sampling. Finally, in Section 4 we show that the finite section method leads to better results in general frame theory than the Casazza–Christensen method.

In the rest of this introduction we collect some basic facts concerning the finite section method. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a separable Hilbert space $\mathcal{H}$ and $\{\mathcal{H}_n\}_{n=1}^{\infty}$ a family of finite-dimensional subspaces of $\mathcal{H}$ for which

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots \subseteq \mathcal{H}_n \uparrow \mathcal{H}. \tag{3}$$

Let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Our purpose is to approximate a bounded operator $V : \mathcal{H} \to \mathcal{H}$ and its inverse. The basic definition, appearing in e.g., [10], is as follows.

Definition 1.1. Let $V : \mathcal{H} \to \mathcal{H}$ be a bounded operator, and assume that for each $n \in \mathbb{N}$ we have given a bounded operator $V_n : \mathcal{H}_n \to \mathcal{H}_n$.

(i) The sequence $\{V_n\}_{n=1}^{\infty}$ is an approximation method for the operator V if

$$V_n P_n f \to Vf \qquad \text{for } n \to \infty \; \forall f \in \mathcal{H}.$$

(ii) An approximation method is applicable if there exists $n_0 \in \mathbb{N}$ such that for all $f \in \mathcal{H}$ the equation

$$V_n x = P_n f \tag{4}$$

has a unique solution x_n for all $n \geqslant n_0$, and x_n converges to a solution of the equation $Vx = f$.

(iii) The sequence $\{V_n\}_{n=1}^{\infty}$ is stable if there exists $n_o \in \mathbb{N}$ such that the operators V_n are invertible on $\mathcal{H}_n$ for $n \geqslant n_0$ and

$$\sup_{n \geqslant n_0} \|V_n^{-1} P_n\| < \infty.$$