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Journal of Approximation Theory 134 (2005) 11 – 64
JOURNAL OF Approximation Theory
www.elsevier.com/locate/jat

Three term recurrence relation modulo ideal and orthogonality of polynomials of several variables

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Received 9 December 2003; received in revised form 20 July 2004; accepted 8 December 2004

Communicated by Walter Van Assche
Available online 14 March 2005

Abstract

Orthogonality of polynomials in several variables with respect to a positive Borel measure supported on an algebraic set is the main theme of this paper. As a step towards this goal quasi-orthogonality with respect to a non-zero Hermitian linear *functional* is studied in detail; this occupies a substantial part of the paper. Therefore necessary and sufficient conditions for quasi-orthogonality in terms of the three term recurrence relation modulo a polynomial ideal are accompanied with a thorough discussion. All this enables us to consider orthogonality in full generality. Consequently, a class of simple objects missing so far, like spheres, is included. This makes it important to search for results on existence of *measures* representing orthogonality on *algebraic sets*; a general approach to this problem fills up the three final sections.

MSC: primary 42C05; 47B25; secondary 47B15

Keywords: Polynomials in several variables; Orthogonal polynomials; Three term recurrence relation; Favard's theorem; Ideal of polynomials; Algebraic set; Symmetric operator; Selfadjoint operator; Joint spectral measure

0. Introduction

Orthogonal polynomials constitute a vital part of Analysis continuously penetrating other areas of Mathematics and much beyond. One of the basic questions of their theory is to decide

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.12.011

whether a sequence of polynomials is orthogonal. In the case of a *single* real variable the most powerful characterization is given by the famous three term recurrence relation. More precisely, orthogonality of a sequence of polynomials with respect to a linear functional reduces immediately the linear dependence of multiplication by the independent variable on these polynomials to a three term relation of recurrence type. The other way, that is from the three term recurrence relation to existence of a linear functional orthogonalizing the sequence of polynomials is established by a result which is commonly known as Favard's Theorem. The important feature of the single variable case is that the linear functional, if positive, is always determined by a measure.

In the *several* variable case the situation is much more complex. Even the meaning of the three term recurrence relation is dubious. The very first difficulty is in finding convenient notation (related to the order in which the orthonormalization procedure has to be performed) which allows us to see the recurrence relation as a three term one. The pioneering attempt in this direction was made by Kowalski [19,20]. A decade later the theme was undertaken by Xu [30–32] and independently by Gekhtman and Kalyuzhny [14,15] (see also [33,34] for further investigations along these lines and [11] for a recent account of the theory).

Further difference is in the fact that the three term recurrence relation may not determine any orthogonality measure though the functional orthogonality in Favard's Theorem is still preserved; regardless the way the relation is built up.

The three term recurrence relation considered in the references alluded to so far forces the Zariski closure of the support of an orthogonalizing measure, provided it exists, to be the whole space $\mathbb{R}^N$ (in fact, this is the only essential case in a single variable). However, important instances, like a sphere, are left out of the game (cf. [11, p. 126]) which calls for extending the study to cases of measures not having too massive support. Our work is intended to get rid of this incompleteness introducing recurrence relations of matrix type satisfied modulo an ideal.

The principal observation is that, if any orthogonality measure exists then the aforesaid ideal consists of all polynomials vanishing on the Zariski closure of its support. The main task is to go the other way around: given an ideal, find (necessary and) sufficient conditions for it to admit measures representing orthogonality of a sequence of polynomials in question. A class of ideals we distinguish for that, called *ideals of type* C, has the property that three term recurrence relations modulo an ideal it induces automatically imply the existence of orthogonalizing measures. However it is not easy to find proper tools to work on this class. Fortunately, the class of ideals of type C contains the cases of algebraic sets of type A and B considered in [24], which can be handled by means of functional analysis and operator theory. This allows us to work out further properties of types A and B, and consequently of type C as well. As a result we get that ideals composed of polynomials vanishing on a compact algebraic set are of type C; other classes considered in this paper correspond to some unbounded algebraic sets. Then basing on a result of [24] we indicate an implicit example of a non-zero proper ideal which is not of type C (we do not know if the zero ideal is of type C). In the case of such ideals we impose some extra conditions (relying on the well know operator theory result of Nelson [21]) on the matrix coefficients appearing in the three term recurrence relation so as to ensure the existence of orthogonalizing measures.

Implementing the programme already described we devote a substantial part of the paper to the so-called *quasi-orthogonality*. It turns out that this notion, due to its simple algebraic