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Generalized polar varieties: geometry and algorithms☆

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Dedicated to Arnold Schönhage

Abstract

Let V be a closed algebraic subvariety of the n-dimensional projective space over the complex or real numbers and suppose that V is non-empty and equidimensional. The classic notion of a polar variety of V associated with a given linear subvariety of the ambient space of V was generalized and motivated in Bank et al. (Kybernetika 40 (2004), to appear). As particular instances of this notion of a generalized polar variety one reobtains the classic one and an alternative type of a polar variety, called *dual*. As main result of the present paper we show that for a generic choice of their parameters the generalized polar varieties of V are empty or equidimensional and smooth in any regular point of V. In the case that the variety V is affine and smooth and has a complete intersection ideal of definition, we are able, for a generic parameter choice, to describe locally the generalized polar varieties of V by explicit equations. Finally, we indicate how this description may be used in order to design in the context of algorithmic elimination theory a highly efficient, probabilistic elimination procedure for the following task: In case, that the variety V is $\mathbb{Q}$-definable and affine, having a complete intersection

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ideal of definition, and that the real trace of V is non-empty and smooth, find for each connected component of the real trace of V an algebraic sample point.
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1. Introduction

The notion of *generalized polar varieties* was introduced and motivated in [4]. The present paper is devoted to the study of their smoothness which entails important algorithmic consequences. It turns out that the classic polar varieties are special instances of the generalized ones. Classic polar varieties were used in [2,3] for the design of a highly efficient elimination procedure which, in the case of an affine, smooth and *compact* real hypersurface or, more generally, a complete intersection variety, produces an algebraic sample point for each connected component of the given real variety. The aim to generalize this algorithmic result to the *non-compact* case motivates the introduction of the concept of the generalized polar varieties of a given algebraic manifold associated with suitably generic linear subvarieties and a non-degenerate hyperquadric of the projective ambient space. In [4] it was shown that these generalized polar varieties become Cohen–Macaulay. This rather *algebraic* than geometric result suffices to solve the algorithmic task which motivates the consideration of generalized polar varieties. In this paper we go a step further: we show a *geometric* result saying that under the same genericity condition the generalized polar varieties of a given algebraic manifold are smooth, and we derive handy local equations for them.

Let $\mathbb{P}^n$ denote the n-dimensional projective space over the field of complex numbers $\mathbb{C}$ and let, for $0 \leqslant p \leqslant n$, V be a pure p-codimensional closed algebraic subvariety of $\mathbb{P}^n$.

Now we are going to outline the basic properties of the notion of a generalized polar variety of V associated with a given linear subspace K, a given non-degenerate hyperquadric Q and a given hyperplane H of the ambient space $\mathbb{P}^n$, subject to the condition that $Q \cap H$ is a non-degenerate hyperquadric of H. We denote this generalized polar variety by $\widehat{W}_K(V)$. It turns out that $\widehat{W}_K(V)$ is empty or a smooth subvariety of V having pure codimension i in V, if V is smooth and K is a "sufficiently generic", $(n-p-i)$-dimensional, linear subspace of $\mathbb{P}^n$, for $0 \leqslant i \leqslant n-p$ (see Corollary 11 and the following comments).

In this paper we consider mainly the case that H is the hyperplane at infinity of $\mathbb{P}^n$, fixing in this manner an embedding of the complex n-dimensional affine space $\mathbb{A}^n$ into the projective space $\mathbb{P}^n$. Let $S := V \cap \mathbb{A}^n$ be the affine trace of V and suppose S is non-empty. Then S is a pure p-codimensional closed subvariety of the affine space $\mathbb{A}^n$.

The affine trace $\widehat{W}_K(S) := \widehat{W}_K(V) \cap \mathbb{A}^n$ is called the *affine* generalized polar variety of S associated with the linear subvariety K and the hyperquadric Q of $\mathbb{P}^n$. The affine generalized polar varieties of S give rise to classic and dual affine polar varieties.

Let us denote the field of real numbers by $\mathbb{R}$ and the real n-dimensional projective and affine spaces by $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{A}^n_{\mathbb{R}}$, respectively. Assume that V is $\mathbb{R}$-definable and let $V_{\mathbb{R}} :=$

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