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Symplectic methods for the approximation of the exponential map and the Newton iteration on Riemannian submanifolds

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Abstract

We use a Hamiltonian approach and symplectic methods to compute the geodesics on a Riemannian submanifold.

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1. Introduction

Let V be a p-dimensional Riemannian real complete manifold. In this paper we study computational aspects of the Newton method for finding zeros of smooth mappings $f : V \to \mathbb{R}^p$. The Newton operator is defined by

$$N_f(x) = \exp_x(-Df(x)^{-1} f(x)). \tag{1}$$

Here $\exp_x : T_x V \to V$ is the exponential map, which "projects" the tangent space at x on the manifold. The Newton method has two important properties: fixed points for N_f correspond to zeros for f and the convergence of the Newton sequences ($x_0 = x$ and $x_{k+1} = N_f(x_k)$) is quadratic for any starting point x in a neighborhood of a nonsingular zero.

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When $V = \mathbb{R}^n$, the exponential map is just a translation: $\exp_x(u) = x + u$ and the Newton operator has the usual form:

$$N_f(x) = x - Df(x)^{-1} f(x)$$

but for a general manifold this is no more true. Except for some cases the exponential map has no analytic expression and we have to compute it numerically: this is the main subject of this paper.

Newton method for maps or vector fields defined on manifolds has already been considered by many authors: Shub [29] defines Newton's method for the problem of finding the zeros of a vector field on a manifold and uses retractions to send a neighborhood of the origin in the tangent space onto the manifold itself. Udriste [35] studies Newton's method to find the zeros of a gradient vector field defined on a Riemannian manifold; Owren and Welfert [27] define Newton iteration for solving the equation $f(x) = 0$ where f is a map from a Lie group to its corresponding Lie algebra; Smith [34] and Edelman et al. [10] develop Newton and conjugate gradient algorithms on the Grassmann and Stiefel manifolds. Shub [30], Shub and Smale [31–33], see also, Blum et al. [4], Malajovich [22], Dedieu and Shub [7] introduce and study the Newton method on projective spaces and their products. Another paper on this subject is Adler et al. [3] where qualitative aspects of Newton method on Riemannian manifolds are investigated for both mappings and vector fields. This paper contains an application to a geometric model for the human spine represented as a 18-tuple of 3×3 orthogonal matrices. Recently, Ferreira–Svaiter [11] give a Kantorovich like theorem for Newton method for vector fields defined on Riemannian manifolds and Dedieu et al. [6] study alpha-theory for both mappings and vector fields.

The computation of the exponential map depends mainly on the considered data structure. In some cases the exponential is given explicitely (Euclidean or projective spaces, spheres …) or may be computed via linear algebra packages (the orthogonal group, Stiefel or Grassmann manifolds [10,34]). The classical description uses local coordinates and the second order system which gives the geodesic curve $x(t)$ with initial conditions $x(0) = x$, and $\dot{x}(0) = u$:

$$\ddot{x}_i(t) + \sum_{j,k} \Gamma^i_{jk} \dot{x}_j(t)\dot{x}_k(t) = 0, \quad 1 \leqslant i \leqslant n,$$

$$x(0) = x, \quad \dot{x}(0) = u.$$

In these equations Γ^i_{jk} are the Christoffel symbols and the exponential is equal to $\exp_x(u) = x(1)$, see Do Carmo [9] or others textbooks on this subject: Dieudonné [8], Gallot et al. [13], Helgason [17]. Such an approach is used by Noakes [25] who considers the problem of finding geodesics joining two given points. We notice that the computation of local coordinates and of the Christoffel symbols may be itself a very serious problem and depends again on the data structure giving the manifold V.

In [5] Celledoni and Iserles consider the approximation of the exponential for finite dimentional Lie groups contained in the general linear group using splitting techniques. Munthe et al. [36] approximate the matrix exponential by the use of a generalized polar decomposition. See also Munthe et al. [36] for the generalized polar decomposition on Lie groups, Krogstad et al. [21] and Iserles et al. [19].