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# Two situations with unit-cost: ordered abelian semi-groups and some commutative rings

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**Abstract**

The paper presents two situations where unit-cost complexity results are closely related with results from the classical computability.

• In Section 2 we study an important theorem by Koiran and Fournier from an axiomatic point of view. It is proved that the algebraic Knapsack problem belongs to P over some ordered abelian semi-group iff P = NP classically. In this case there would exist a unit-cost machine solving the algebraic Knapsack problem over all ordered abelian semi-groups in some uniform polynomial time.

• In Section 3 we apply the theorem of Matiyasevich in order to construct a ring with P $\neq$ NBP $\neq$ NP and such that its polynomial hierarchy does not collapse at any level.


*MSC:* 03C60; 03D15

*Keywords:* Computation; Unit-cost; Bit-cost; P vs. NP; Ordered abelian semi-group; Axioms; Products of rings; Polynomial hierarchy; Arithmetic hierarchy

## 1. Introduction

This paper presents two situations where unit-cost complexity results are closely related with results from the classical computability. In Section 2 we study an important theorem by Koiran and Fournier from an axiomatic point of view.

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**Definition.** The classical problem Knapsack: given a finite list of natural numbers, it is asked if the last number in the list is a sum of some other numbers in the list. There are many other versions and specifications and many ways to encode the Knapsack problem. The present version is sometimes also called Subset Sub and denoted SSS. All variants of Knapsack are known to be NP-complete in the classical sense, see [6,9,15,22].

**Definition.** The algebraic Knapsack problem (Knapsack with unit cost): given a finite list of real numbers (or, more generally, a list of elements in an algebraic structure with addition) it is asked if the last element in the list is a sum of some other elements in the list.

The following example should stress the differences between Knapsack as a classical (Turing) problem and Knapsack as an algebraic (unit-cost) problem: An instance of the classical problem Knapsack is for example a word of length $n$ with letters in the alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \natural\}$ like for example:

$$15674\natural08\natural900\natural666\natural0002\natural15\natural010.$$

This instance has length 28 and is a solution of the problem. If we look at this instance as an instance of the algebraic unit-cost Knapsack problem, (over some structure with addition, like $(\mathbb{R}, +, <)$ or like $(\mathbb{N}, +, \cdot)$) then its length is only 7. The problem to algorithmically decide in a polynomial time in the digit-cost if an instance is a solution seems at the very first sight to be very different from the same problem to be solved in polynomial time according to the unit-cost. Also, the fact that in the first case just a recursive function in words must be computed and that in the second case one must use only functions, relations and constants given in the structure contributes to this belief.

Nevertheless Fournier and Koiran proved in [4] the following surprising result:

**Theorem 1.1.** *The ordered additive group of the reals* $(\mathbb{R}, +, -, <, 0, 1)$ *has* $\mathrm{P} = \mathrm{NP}$ *in the sense of unit-cost* (*algebraic*) *complexity for parameter-free computations if and only if* $\mathrm{P} = \mathrm{NP}$ *in the classical sense*.

For proving their result, Fournier and Koiran refined older geometric constructions by Meyer auf der Heide, see [13]. The most important notion in this inductive proof is maybe auf der Heide's invariant called *coarseness* of a hyperplane arrangement. The coarseness is the maximal radius of a hyperball with the following property: if the ball meets at least two hyperplanes in this arrangement, then they are not parallel and the ball intersects also their common intersection. In the proof of the theorem one can consider a problem in the sense of unit-cost (algebraic) complexity over the ordered reals with addition as a family of arrangements of polynomials in different dimensions. Without restricting the generality we can suppose the input to be in an unitary hypercube. This hypercube is partitioned in small hypercubes having a radius smaller than the coarseness of the hyperplane arrangement representing the problem in the given dimension. The coarseness is a rational number which can be computed with a method given by Meyer auf der Heide: to represent it binary takes a length bounded by a polynomial in the parameter $n$ (which is in the same time unit-cost of the input). Localizing the input in some small box is done by binary search. The geometrical situation in the small box is projected on a $n-1$ dimensional side of the original hypercube,