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An intrinsic homotopy for intersecting algebraic varieties

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Abstract

Recently we developed a diagonal homotopy method to compute a numerical representation of all positive dimensional components in the intersection of two irreducible algebraic sets. In this paper, we rewrite this diagonal homotopy in intrinsic coordinates, which reduces the number of variables, typically in half. This has the potential to save a significant amount of computation, especially in the iterative solving portion of the homotopy path tracker. Three numerical experiments all show a speedup of about a factor two.

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0. Introduction

Our goal is to compute the irreducible decomposition of $A \cap B \subset \mathbb{C}^k$, where A and B are irreducible algebraic sets. In particular, suppose that

- A is an irreducible component of the solution set of a polynomial system $f_A(u) = 0$ defined on $\mathbb{C}^k$, and similarly
- B is an irreducible component of the solution set of a polynomial system $f_B(u) = 0$ defined on $\mathbb{C}^k$.

This includes the important special case when f_A and f_B are the same system, but A and B are distinct irreducible components.

Casting this problem into the framework of numerical algebraic geometry, we assume that all components are represented as *witness sets*. For an irreducible component $A \subset \mathbb{C}^k$ of dimension $\dim(A)$ and degree $\deg(A)$, a witness set consists of a generic $k - \dim(A)$ dimensional linear subspace $L \subset \mathbb{C}^k$ and the $\deg(A)$ points of intersection $A \cap L$. We assume that at the outset we are given such sets for A and B, and our goal is to compute witness sets for the irreducible components of $A \cap B$. The intersection may break into several such components, and the components may have various dimensions. Our methods proceed in two phases: we first find a witness superset guaranteed to contain witness points for all the components, then we break this set into its irreducible components. We recently reported on an algorithm [15], herein called the *extrinsic*[3] homotopy method, for computing a witness superset for $A \cap B$. This can then be decomposed into irreducible components using the methods in [14] and its references.

Abstracting away the details, which are discussed more fully in Section 1, the extrinsic method consists of a cascade of homotopies in unknowns $x \in \mathbb{C}^N$ and path parameter $t \in [0, 1]$, each of the form

$$H(x,t) := \begin{bmatrix} f(x) \\ t(Px+p) + (1-t)(Qx+q) \end{bmatrix} = 0, \tag{1}$$

where $f : \mathbb{C}^N \to \mathbb{C}^m$ is a system of polynomial equations, P, Q are $(N-m) \times N$ full-rank matrices, and $p, q \in \mathbb{C}^{(N-m)}$ are column vectors. There is a homotopy of this form for each dimension where $A \cap B$ could have one or more solution components. We know solution values for x at $t = 1$ and wish to track solution paths $x(t)$ implicitly defined by (1) as $t \to 0$ to get $x(0)$.

At any specific value of t, this looks like

$$\widehat{H}(x,t) = \begin{bmatrix} f(x) \\ R(t)x + r(t) \end{bmatrix} = 0, \tag{2}$$

where $R = tP + (1-t)Q$ and $r = tp + (1-t)q$. The homotopy is constructed such that we are assured that $R(t)$ is full rank for all $t \in [0, 1]$. Thus, the linear subspace of solutions of $R(t)x + r(t) = 0$ can be parameterized by $u \in \mathbb{C}^m$ in the form

$$x(u,t) = R^{\perp}(t)u + x_p(t), \tag{3}$$

[3] The terminology extrinsic/intrinsic is in analogy with the homotopies of [4].