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Approximation on anisotropic Besov classes with mixed norms by standard information☆

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Abstract

This article considers the approximation problem on periodic functions of anisotropic Besov classes with mixed norms using standard information. The asymptotic decay rates of the best algorithms in the worst-case setting are determined. An interpolating algorithm that attains this decay rate is given as well.

Keywords: Information-based complexity; Exact order; Interpolation; Optimal recovery

1. Introduction and main results

In the 1950s, under the influence of the work Kolmogorov [9], a new perspective on approximation theory developed. Classical approximation theory had already accumulated

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a huge amount of results on the approximation by algebraic and trigonometric polynomials. Also, non-classical tools for approximation, such as splines, had began to permeate computational practice. Therefore, it was natural to ask how to compare different methods of approximation, how to determine an optimal method, and how to construct optimal or nearly-optimal algorithms in various settings. This led to the notions of widths, optimal recovery, and computational complexity. There have been many beautiful results on the exact asymptotic orders of these quantities for various function spaces, but there still remain many important open problems.

The extensive literature devoted to the widths, optimal recovery of functions, and computational complexity, includes the work of Heinrich [5], Micchelli and Rivlin [15,16], Novak [19], Pinkus [20], Ritter [21], Traub et al. [30] and Traub and Woźniakowski [31]. In particular, Heinrich [5] and Luo and Sun [12] have investigated the weak asymptotic order of the reconstruction of functions in classical Sobolev spaces using their values at n points. Kudryavtsev [10,11] has studied the same problem for non-periodic isotropic Besov spaces. His method to obtain upper bounds is different from the method used in this article.

It is well known that approximation plays a dominant role in the class of linear multivariate problems, and the results on approximation can be often used for other multivariate problems including integration. This article studies an approximation problem using standard information, that is the values of the function at selected points. Specifically, the function to be approximated lies in a multivariate anisotropic Besov space of periodic functions $B^{\mathbf{r}}_{\mathbf{p}\theta}$ defined below.

In this article $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. Moreover, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_+$ the set of non-negative integers, and $\mathbb{N}$ the set of positive integers. The notation $\mathbb{R}^d$ denotes the space of d-dimensional vectors, and analogously for $\mathbb{R}^d_+$, etc. The periodic functions to be approximated are defined on the d-dimensional torus, $\mathbb{T}^d := [0, 2\pi]^d$.

Let $f(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{T}^d$, be a measurable, almost everywhere finite real function which is 2π-periodic in each variable. This function is in $L_{\mathbf{p}}(\mathbb{T}^d)$, for $\mathbf{p} = (p_1, p_2, \ldots, p_d)$, $1 \leqslant p_j < \infty$, $j = 1, 2, \ldots, d$, if

$$\|f\|_{\mathbf{p}} := \left(2\pi \int_0^{2\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\mathbf{x})|^{p_1}\, dx_1\right)^{p_2/p_1} dx_2 \right.\right.$$
$$\left.\left.\cdots\right)^{p_d/p_{d-1}} dx_d\right)^{1/p_d} < \infty,$$

with the usual modification if some p_i are infinite, and the notation $L^{\mathbf{s}}_{\mathbf{p}}(\mathbb{T}^d)$ represents the space of all functions $f \in L_{\mathbf{p}}(\mathbb{T}^d)$ whose s_j partial derivatives $\partial^{s_j} f/\partial x_j^{s_j}$ on variate x_j, $j = 1, \ldots, d$ are also in $L_{\mathbf{p}}(\mathbb{T}^d)$. The mixed norm [25, p. 21] is given by

$$\|f\|_{L^{\mathbf{s}}_{\mathbf{p}}} := \|f\|_{\mathbf{p}} + \sum_{j=1}^{d} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \right\|_{\mathbf{p}},$$