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Existence, uniqueness and stability of travelling waves in a discrete reaction–diffusion monostable equation with delay☆

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Abstract

In this paper, we study the existence, uniqueness and asymptotic stability of travelling wavefronts of the following equation:

$$u_t(x,t) = D[u(x+1,t) + u(x-1,t) - 2u(x,t)] - du(x,t) + b(u(x,t-r)),$$

where $x \in \mathbb{R}$, $t > 0$, $D, d > 0$, $r \geqslant 0$, $b \in C^1(\mathbb{R})$ and $b(0) = dK - b(K) = 0$ for some $K > 0$ under monostable assumption. We show that there exists a minimal wave speed $c^* > 0$, such that for each $c > c^*$ the equation has exactly one travelling wavefront $U(x+ct)$ (up to a translation) satisfying $U(-\infty) = 0, U(+\infty) = K$ and $\limsup_{\xi \to -\infty} U(\xi)e^{-\Lambda_1(c)\xi} < +\infty$, where $\lambda = \Lambda_1(c)$ is the smallest solution to the equation $c\lambda - D[e^{\lambda} + e^{-\lambda} - 2] + d - b'(0)e^{-\lambda c r} = 0$. Moreover, the travelling wavefront is strictly monotone and asymptotically stable with phase shift in the sense that if an initial data $\varphi \in C(\mathbb{R} \times [-r, 0], [0, K])$ satisfies $\liminf_{x \to +\infty} \varphi(x, 0) > 0$

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and $\lim_{x\to-\infty}\max_{s\in[-r,0]}|\varphi(x,s)e^{-\Lambda_1(c)x}-\rho_0 e^{\Lambda_1(c)cs}|=0$ for some $\rho_0\in(0,+\infty)$, then the solution $u(x,t)$ of the corresponding initial value problem satisfies $\lim_{t\to+\infty}\sup_{\mathbb{R}}|u(\cdot,t)/U(\cdot+ct+\xi_0)-1|=0$ for some $\xi_0=\xi_0(U,\varphi)\in\mathbb{R}$.

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1. Introduction

Travelling wavefront solutions play an important role in describing the long-term behaviour of solutions to initial value problems in reaction and diffusion (both continuous and discrete) equations. Such solutions also have their own practical background, such as, transition between different states of a physical system, propagation of patterns, and domain invasion of species in population biology. When the nonlinear reaction term is of *monostable type*, that is, considering the R-D equation

$$w_t(x,t)=Dw_{xx}(x,t)+f(w(x,t)),\quad x\in\mathbb{R},\ t\geqslant 0, \tag{1.1}$$

with $f(w)$ satisfying

(A) $f(0)=f(k)=0$ for some $k>0$; and $f(w)>0$ for $w\in(0,k)$,

it has been known from long time ago that $c_{\min}=2\sqrt{Df'(0)}>0$ is the minimal wave speed in the sense that (i) for every $c>c_{\min}$ there exists a travelling wavefront of the form $w(x,t)=u(x+ct)$ with $u(s)$ increasing and $u(-\infty)=0$, $u(\infty)=k$; (ii) the wavefront is unique up to translation; (iii) for $c<c_{\min}$, there is no such monotone wavefront with speed c. Moreover, the wavefront cannot be stable with respect to general initial functions, it can, however, be stable in respect to some smaller class of initial functions (e.g., initial functions with compact support).

For a spatially discrete analogue of (1.1), one may consider the following lattice differential equations

$$u_n'(t)=D[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)]+f(u_n(t)),\quad n\in\mathbb{Z},\ t>0. \tag{1.2}$$

System (1.2) can either be considered as a discretization of (1.1), or be derived directly from population models over patchy environments (see, e.g., [3,12,18]). Indeed, as mentioned in Bell and Cosner [3] and Keener [12], in many situations, one usually derives a discrete version like (1.2) first, and then, by taking limit, arrives at a continuous version like (1.1). When the nonlinear term in (1.2) is of *bistable type*, the study on travelling wavefronts of such lattice differential equations have been extensive and intensive, and has resulted in many interesting and significant results, some of which, have revealed some essential difference between a discrete model and its