Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 218 (2005) 15–35

Journal of Differential Equations

www.elsevier.com/locate/jde

Multiplicity results for periodic solutions to delay differential equations via critical point theory☆

Zhiming Guo, Jianshe Yu*

Department of Applied Mathematics, Guangzhou University, Guangzhou 510405, People's Republic of China

Received 20 January 2004; revised 25 July 2005

Available online 21 September 2005

Abstract

By the critical point theory, we study the existence and multiplicity of periodic solutions to the following system of delay differential equations:

$$x'(t) = -f(x(t-r)), \tag{$*$}$$

where $x \in \mathbf{R}^n$, $f \in C(\mathbf{R}^n, \mathbf{R}^n)$, and $r > 0$ is a given constant. A sufficient condition on the existence and multiplicity of periodic solutions for Eq. $(*)$ is obtained. To the authors' knowledge, this is the first time, the existence of periodic solutions to systems of delay differential equations is dealt with by using variational approaches directly.

MSC: 34K13; 34K18; 58E50

Keywords: Delay differential equations; Multiple periodic solutions; Critical point theory; Pseudo-index

☆ This project was supported by National Natural Science Foundation of China (No. 10471029), Natural Science Foundation of Hunan Province (No. 03JJY4001) and the Foundation of First Period of Key Basic Research Sponsored by the Department of Science and Technology of China (2003CCA02400).
* Corresponding author.

E-mail addresses: gzm100@21cn.com (Zhiming Guo), jsyu@gzhu.edu.cn (Jianshe Yu).

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2005.08.007

1. Introduction

In this paper, we are concerned with the existence of periodic solutions to the system of delay differential equations

$$x'(t) = -f(x(t-r)), \tag{1.1}$$

where $x \in \mathbf{R}^n$, $f \in C(\mathbf{R}^n, \mathbf{R}^n)$, and $r > 0$ is a given constant.

For the case $n = 1$, there has been a long history in the study of the existence of periodic solutions to (1.1). To the authors' knowledge, the earliest result was obtained by G.S. Jones in 1962. In his paper [14], Jones gave an existence result on the periodic solutions to the following equation:

$$u'(t) = -au(t-1)[1 + u(t)]. \tag{1.2}$$

One can easily find that (1.2) can be changed to (1.1) by letting $1 + u = e^x$. Jones's idea was to find a cone that maps into itself under the flow defined by (1.2), and then to use some fixed point theorems to determine the periodic solutions. However, in most cases, zero is an equilibrium point. Therefore one is interested in finding nonzero fixed points of a mapping of a cone. To this end, a number of fixed point theorems for cone mappings (see for example [9]) as well as mappings of a convex set into itself with an ejective fixed point as an extreme point of the convex set (see [3]) have been established. By using the idea of Browder [3], Nussbaum obtained a series of existence theorems (see [27,28]).

In 1970s of the last century, there were still many other effective methods developed to investigate the existence of periodic solutions to (1.1) and to even more general delay differential equations of the form

$$x'(t) = f(x_t). \tag{1.3}$$

For example, the Hopf bifurcation theorem for delay differential equations first introduced by Chow and Mallet-Paret can be effectively used to study the periodic solutions near an equilibrium point. The global Hopf bifurcation theorem was obtained by Chow and Mallet-Paret [5] and Nussbaum [29]. Coincidence degree theory introduced by Mawhin [24,25] also proved to be a powerful tool in studying the existence problem for delay differential equations.

In 1974, Kaplan and Yorke [15] introduced another technique studying the existence of periodic solutions of (1.1) when $n = 1$ that may reduce the existence problem of periodic solutions of (1.1) to a problem finding periodic solutions of an associated plane ordinary differential system.

The Poincaré–Bendixson theorem can also be applied to find the periodic solutions for (1.1). Results in this direction began with early work of Kaplan and Yorke [16,17] and were given by several authors. But most of these results concern scalar equations, and generally slowly oscillating solutions. In 1996, by using a discrete (integer-valued)