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Existence and continuation of periodic solutions of autonomous Newtonian systems

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Abstract

In this article, we study the existence and the continuation of periodic solutions of autonomous Newtonian systems. To prove the results we apply the infinite-dimensional version of the degree for $SO(2)$-equivariant gradient operators defined by the third author in Nonlinear Anal. Theory Methods Appl. 23(1) (1994) 83–102 and developed in Topol. Meth. Nonlinear Anal. 9(2) (1997) 383–417. Using the results due to Rabier [Symmetries, Topological degree and a Theorem of Z.Q. Wang, J. Math. 24(3) (1994) 1087–1115] and Wang [Symmetries and calculation of the degree, Chinese Ann. Math. 10 (1989) 520–536] we show that the Leray–Schauder degree is not applicable in the proofs of our theorems, because it vanishes.

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1. Introduction

The first aim of this article is to study the existence of periodic solutions of the following system:

$$\ddot{x} = -V'(x), \tag{1.1}$$

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and V' denotes the gradient of V. We assume that $(V')^{-1}(0) = \{p_1, \ldots, p_q\}$ is a finite set and that $V'(x) = V''(\infty) \cdot x + o(\|x\|)$ as $\|x\| \to \infty$, where $V''(\infty)$ is a real symmetric $(n \times n)$-matrix.

Such a problem has been considered for $q = 1$ by Amann and Zehnder, see [2], and by Benci and Fortunato, see [5], for any $q \in \mathbb{N}$.

Benci and Fortunato estimated the number of non-stationary T-periodic solutions of (1.1) as $T \to \infty$. To avoid some technicalities and to make the proofs more transparent they assumed that all the non-stationary T-periodic solutions are not T-resonant and that potential V is a Morse function. These assumptions seem to be restrictive and rather difficult to verify.

We relax these assumptions and therefore we obtain only the existence of at least one non-stationary T-periodic solution of (1.1). We formulate the sufficient conditions for the existence of non-stationary T-periodic solutions of (1.1) in terms of $V''(p)$ and $\mathrm{ind}(-V', p)$, where $p \in \{p_1, \ldots, p_q, \infty\}$. It is worth to point out that we can treat problems with resonance at stationary solutions and at the infinity. As a basic tool we use the degree for $SO(2)$-equivariant gradient maps, see [23,24].

The second aim of this article is to study the continuation of non-stationary T-periodic solutions of the following system:

$$\ddot{x} = -V'_\lambda(x), \tag{1.2}$$

where $V_\lambda \in C^2(\mathbb{R}^n, \mathbb{R})$ for $\lambda \in \mathbb{R}$ and potential V_0 possesses all the properties of potential V in (1.1). We formulate sufficient conditions for the existence of connected sets of T-periodic solutions of (1.2) emanating from level $\lambda = 0$.

We consider solutions of (1.1) and (1.2) as critical points of $SO(2)$-invariant functional defined on a suitably chosen Hilbert space, which is an orthogonal representation of the group $SO(2)$, see also [18,20]. Gradient of this functional is an $SO(2)$-equivariant map in the form of a compact perturbation of the identity.

It is known that the Conley index and the Morse theory are not suitable tools for the study of global bifurcations and the continuation of critical points of functionals, see [3,6,14,17,25] for discussion and examples. Since considered gradient is $SO(2)$-equivariant, the Leray–Schauder degree is not applicable in our approach because it vanishes, see [19,26] and Remark 5.2.6. Therefore to prove our results we apply the degree for $SO(2)$-equivariant gradient maps. Degree for G-equivariant gradient maps has been defined in [8] for $G = SO(2)$. Next it was improved in [23] and in [12] for symmetries of any compact Lie group G.