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Analyticity of classical steady needle crystals

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Abstract

This paper is a continuation of our previous work "Rigorous results in selection of steady needle crystals, J. Differential Equations 197 (2004) 349–426". It concerns analyticity of a classical steadily translating needle crystal. It is proved that any classical solution to the needle crystal problem with sufficiently small but nonzero surface tension, if its slope deviation is close to some Ivantsov zero-surface-tension solution and if its curvature satisfies some algebraic decay conditions at ∞, must belong to the analytic function space $\mathbf{A}_0$ defined in §1 and chosen in the previous study mentioned above. The analyticity result implies that there can be no classical steady needle crystal solution when anisotropy is zero.

Keywords: Steady needle crystal; Analytic solution; Analytic continuation; Integro-differential equation

1. Introduction and notations

Dendritic crystal growth has long been a subject of continued investigations. Reviews of the subject from various perspectives can be found in [13,20]. The simplest example of dendrite growth is the growth of a needle crystal in solidification from a pure undercooled melt. For zero surface tension, Ivantsov [11] found an infinite continuous family of parabolic steadily growing crystals without side branching (called a needle crystal). These Ivantsov solutions do not produce a unique steady dendrite velocity U and the radius of curvature of the tip a, as experimental evidence suggests, but rather determine the product of the tip radius of curvature and the steady dendrite

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velocity. To be specific, Ivantsov's solution produces only a single relation between the dimensionless undercooling $\Delta = \frac{c_p(T_m - T_\infty)}{L}$ and the Peclet number $P = \frac{Ua}{2D}$. (In these formulas, c_p is the heat capacity, L is the latent heat, T_m is the melting temperature, T_∞ is the specific temperature at infinity and D is the thermal diffusivity.) The reason is that the Ivantsov problem is missing a length scale. The only quantities in the theory with dimensions of length are the tip radius a and the diffusion length $l = 2D/U$; thus, one dimensionless relationship between $P = a/l$ and Δ is all that can be expected.

This dimensional degeneracy of the Ivantsov problem suggests that capillarity is an essential physical ingredient for the dendritic selection mechanism. A new length scale associated with the surface tension d is chosen to be $d_0 = \frac{dc_p}{aL} T_m$. When surface tension is taken into account, there is enough dimensional information to determine dendrite velocity and tip curvature in terms of undercooling. However, this need not imply that a solution exists in this case. Kruskal and Segur [16] studied the third-order differential equations arising from one of the phenomenological models. They proved that in the limit of zero surface tension, these equations from the geometric model of growing dendrites do not have any physically acceptable solutions when crystalline anisotropy is ignored even though the equations admit solutions when surface tension is zero. This extraordinary situation happens due to the effect of exponential terms in an asymptotic expansion for small surface tension. When crystalline anisotropy is included in the geometric model, a discrete set of solution is found to exist. Based on different models (including Nash–Glicksman equation [24]), these conclusions were supported by the numerical work of Kessler and Levine [14] and by formal analytical calculations of Pelce and Pomeau [27], Ben Amar and Pomeau [5], Barbeiri et al. [1] and Tanveer [29].

There have been several models for dendritic growth and solidification problems in the literature. Besides work based on the geometric model and the Nash–Glicksman model mentioned above, there are also numerous works based on the phase field model and the sharp interface model (see [6] and references cited therein). In this paper, our analysis is based on a one-sided model as in [1,17,34]. In this one-sided model, the heat diffusion in the solid phase is neglected. The dimensionless temperature T satisfies the heat diffusion equation in the liquid region. A far-field condition on temperature is specified as well in accordance with the experimental condition. On the free boundary, one specifies two interfacial boundary conditions: one is the Gibbs–Thompson boundary condition that accounts for lowering of the melting temperature by curvature, while the other follows from a balance of heat at the interface. Considering a conformal map $z(\xi)$ that maps the upper-half ξ-plane into the physical region (liquid region) in the z plane. The real ξ-axis corresponds to the unknown interface. It is clear that determination of the function $z(\xi)$ yields the unknown interface. We decompose $z(\xi)$ into

$$z(\xi) = -\frac{i}{2}\,\xi^2 + \xi + F(\xi),$$

where $z_I(\xi) = -\frac{i}{2}\,\xi^2 + \xi$ is the Ivantsov solution on ξ plane. Following [34], a steady symmetric needle crystal is equivalent to finding function F analytic in the upper-