Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

J. Differential Equations 211 (2005) 303–332

Journal of Differential Equations

www.elsevier.com/locate/jde

Decay of solutions of the wave equation with arbitrary localized nonlinear damping

Mourad Bellassoued

Départment des Mathematiques, Faculté des sciences de Bizerte, 7021 Jarzouna, Bizerte, Tunisia

Received 1 December 2003; revised 5 August 2004

Abstract

We study the problem of decay rate for the solutions of the initial–boundary value problem to the wave equation, governed by localized nonlinear dissipation and without any assumption on the dynamics (i.e., the control geometric condition is not satisfied). We treat separately the autonomous and the non-autonomous cases. Providing regular initial data, without any assumption on an observation subdomain, we prove that the energy decays at last, as fast as the logarithm of time. Our result is a generalization of Lebeau (in: A. Boutet de Monvel, V. Marchenko (Eds.), Algebraic and Geometric Methods in Mathematical Physics, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1996, pp. 73) result in the autonomous case and Nakao (Adv. Math. Sci. Appl. 7 (1) (1997) 317) work in the non-autonomous case. In order to prove that result we use a new method based on the Fourier–Bross–Iaglintzer (FBI) transform.

Keywords: Decay rate; Initial–boundary value problem; Wave equation; FBI transform

1. Introduction

The aim of this paper is to give decay estimates for the wave equation with a nonlinear damping term and without any assumption on the dynamics. Let Ω be a smooth n-dimensional Riemannian compact manifold with boundary $\Gamma = \partial\Omega$. We start an investigation of the asymptotic behavior of the solution of the following

E-mail address: mourad.bellassoued@fsb.rnu.tn.

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2004.12.010

wave equation:

$$\partial_t^2 u - \Delta u + b(t,x)g(\partial_t u) = 0 \quad \text{in} \quad Q = \Omega \times \mathbb{R}^+, \tag{1.1}$$

with the Dirichlet boundary condition:

$$u = 0 \quad \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \tag{1.2}$$

where $b(t,x)$ is given by

$$b(t,x) = (1+t)^\theta a(x) := \sigma(t)a(x), \quad -1 < \theta \leqslant 0, \tag{1.3}$$

Δ is the Laplace–Beltrami operator on Ω, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function with $g(0) = 0$, $sg(s) \geqslant 0$ and $a \in L^\infty(\Omega)$ is assumed to be a positive function $a(x) \geqslant 0$ for all $x \in \Omega$. Let $\omega \subset\subset \Omega$ be a given arbitrary non-empty subdomain such that $a(x) \geqslant a_0 > 0$ in $\omega \subset\subset \Omega$. We are interested in a *semi-dynamical system* associated with (1.1) and (1.2). Let us take the product-space $X = H_0^1(\Omega) \oplus L^2(\Omega)$, where the norm in $H_0^1(\Omega)$ is defined by

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}, \quad v \in H_0^1(\Omega). \tag{1.4}$$

The norm in X is chosen as follows:

$$\|(v,w)\|_X^2 = E(v,w) = \|v\|^2_{H_0^1(\Omega)} + \|w\|^2_{L^2(\Omega)}, \quad \text{for} \quad (v,w) \in X. \tag{1.5}$$

It is known that (1.1) and (1.2) define an evolution in X in a natural way: any initial state $u = (u_0, u_1) \in X$ will transform in time into the state $(u(t), \partial_t u(t))$, with the initial conditions

$$u(0,x) = u_0, \quad \partial_t u(0,x) = u_1. \tag{1.6}$$

Thus, from the very beginning we have to impose certain restrictions on g in order to guarantee the global existence, uniqueness and continuous dependence on the initial data. We will assume that $g(s)$ satisfies the following conditions:

(i) There exists $C_1, C_2 > 0$ and $r \geqslant 1$ such that for $|s| \leqslant 1$, we have

$$C_1 |s|^r \leqslant |g(s)| \leqslant C_2 |s|^{\frac{1}{r}}. \tag{1.7}$$

(ii) There exists $C_1', C_2' > 0$ such that for $|s| > 1$ we have

$$C_1' |s|^k \leqslant |g(s)| \leqslant C_2' |s|^p,$$