Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Differential Equations 217 (2005) 257–281

Journal of Differential Equations

www.elsevier.com/locate/jde

Mountain pass solutions for singularly perturbed nonlinear Dirichlet problems☆

Jaeyoung Byeon*

Department of Mathematics, POSTECH, Pohang, Kyungbuk 790-784, Republic of Korea

Received 22 June 2004; revised 18 July 2005

Abstract

We consider the following singularly perturbed nonlinear elliptic problem

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where the nonlinearity f is of subcritical growth and Ω is a bounded domain in $\mathbf{R}^n$. Under certain conditions of f, there exists a mountain pass solution u_ε for above problem and the solution u_ε exhibits a spike layer as $\varepsilon \to 0$. To see the location of the spike layer for small $\varepsilon > 0$, some additional assumptions, certain nondegeneracy for a limiting problem or monotonicity of $f(t)/t$, have been assumed. In this paper, we characterize the location of the spike layer without such additional assumptions for $n \geqslant 3$.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with $\partial\Omega \in C^2$. We are interested in the following singularly perturbed nonlinear elliptic problem on Ω

$$\varepsilon^2\Delta u - u + f(u) = 0, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

☆ This work was supported by Grant No. R01-2004-000-10055-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

* Fax: +82 54 279 2799.
E-mail address: jbyeon@postech.ac.kr.

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2005.07.008

A solution of (1) corresponds to a critical point of a functional $\Gamma^\varepsilon(u) = \frac{1}{2}\int_\Omega \varepsilon^2|\nabla u|^2 + u^2\,dx - \int_\Omega F(u)\,dx$ on $H_0^{1,2}(\Omega)$. Here, $F(t) = \int_0^t f(s)\,ds$. The following equation will correspond to a limiting equation to Eq. (1) as $\varepsilon \to 0$;

$$\Delta u - u + f(u) = 0, \quad u > 0 \text{ in } \mathbf{R}^n \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{2}$$

We assume that a function $f : \mathbf{R} \to \mathbf{R}$ satisfies the following conditions:

(f1) $f \in C(\mathbf{R},\mathbf{R})$, $f(t) = 0$ for $t \leqslant 0$ and $\lim_{t\to 0} f(t)/t = 0$;
(f2) there exists $p \in \left(1, \frac{n+2}{n-2}\right)$ such that $\limsup_{t\to\infty} f(t)/t^p < \infty$;
(f3) for $F(t) \equiv \int_0^t f(s)\,ds$, there exists $\mu > 2$ such that $0 < \mu F(t) < f(t)t$ for $t > 0$.

Under conditions (f1)–(f3), there exists a mountain pass solution v_ε of (1) which is positive on Ω. Then, we can deduce from comparison principles that for a maximum point x_ε of v_ε, there exist constants $C, c > 0$, independent of $\varepsilon > 0$ satisfying $v_\varepsilon(x) \leqslant C \exp\left(-\frac{c\,\mathrm{dist}(x,x_\varepsilon)}{\varepsilon}\right)$, $x \in \Omega$. Thus, v_ε exhibits a spike layer as $\varepsilon \to 0$. Then, a natural concern is the location of the maximum points x_ε of the solution v_ε for small $\varepsilon > 0$. In a striking paper [12], Ni and Wei proved that

$$\lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \partial\Omega) = \max_{x\in\Omega} \mathrm{dist}(x, \partial\Omega) \tag{3}$$

under the following additional conditions

(f1′) $f \in C^{0,1}(\mathbf{R},\mathbf{R})$;
(f4) $f(t)/t$ is non-decreasing on $(0,\infty)$;
(f5) for a least energy solution U of (2), if $\Delta V - V + f'(U)V = 0$ and $V \in H^{1,2}(\mathbf{R}^n)$, then, $V = a_1 \frac{\partial U}{\partial x_1} + \cdots + a_n \frac{\partial U}{\partial x_n}$ for some $a_1, \ldots, a_n$.

In an interesting paper [5], del Pino and Felmer showed that the asymptotic behaviour (3) can be obtained in a simple manner even without conditions (f1′) and (f5). Their approach in [5] depends strongly on the monotonicity condition (f4). In this paper, developing further the approach in [5], we will prove the asymptotic behaviour (3) in the case of $n \geqslant 3$ without condition (f4). Thus, we need just conditions (f1)–(f3) to show the asymptotic behaviour (3). One of basic ideas in [5] is to consider a pass $tu, t \in (0,\infty)$ for each $u \in H_0^{1,2}(\Omega)$; a function $g(t) = \Gamma^\varepsilon(tu)$ has only maximum critical points on $(0,\infty)$ when (f4) holds. We can say that a pass $\{tu \mid t \in [0,\infty)\}$ is made by deforming the range of u. A main idea or a different view point with [5] in this paper is to deform the domain of u instead of the range of u.

This paper is organized as follows. In Section 2, defining some necessary terms, we state our main result. Then, we prepare some preliminary results for the proof of the main result in Section 3. Lastly, in Section 4, we prove our main result.