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Energy estimate and fundamental solution for degenerate hyperbolic Cauchy problems

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Abstract

The aim of this paper is to give an uniform approach to different kinds of degenerate hyperbolic Cauchy problems. We prove that a weakly hyperbolic equation, satisfying an intermediate condition between effective hyperbolicity and the C^∞ Levi condition, and a strictly hyperbolic equation with non-regular coefficients with respect to the time variable can be reduced to first-order systems of the same type. For such a kind of systems, we prove an energy estimate in Sobolev spaces (with a loss of derivatives) which gives the well-posedness of the Cauchy problem in C^∞. In the strictly hyperbolic case, we also construct the fundamental solution and we describe the propagation of the space singularities of the solution which is influenced by the non-regularity of the coefficients with respect to the time variable.

Keywords: Degenerate hyperbolic equations; Energy estimates; Propagation of singularities

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1. Introduction

Let us consider the Cauchy problem

$$\begin{cases} P(t,x,D_tD_x)u(t,x)=0, & (t,x)\in]-T,T[\times\mathbf{R}^n, \\ u(0,x)=u_0(x), \\ \partial_t u(0,x)=u_1(x) \end{cases} \tag{1.1}$$

for the second-order operator

$$\begin{cases} P=D_t^2-a(t,x,D_x)+b(t,x,D_x)+c(t,x), \\ a(t,x,\xi)=\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j, \\ b(t,x,\xi)=\sum_{j=1}^{n} b_j(t,x)\xi_j, \end{cases} \tag{1.2}$$

$D=\frac{1}{\sqrt{-1}}\partial$, under the hyperbolicity condition

$$a(t,x,\xi)\geqslant 0, \quad t\in]-T,T[,x,\xi\in\mathbf{R}^n. \tag{1.3}$$

Concerning the regularity of the coefficients, we assume $a_{ij}(t,\cdot)\in\mathcal{B}^\infty(\mathbf{R}^n)$, $t\in]-T,T[$, $b_j,c\in\mathcal{B}^0(]-T,T[;\mathcal{B}^\infty(\mathbf{R}^n))$, $\mathcal{B}^k(\Omega;Y)$ the space of all functions from Ω to Y which are bounded together with all their derivatives up to the order k. The regularity of the a_{ij}'s with respect to the time variable t will be specified from case to case.

We say that the Cauchy problem (1.1) is well-posed in the space X of functions in $\mathbf{R}^n$ if for every $u_0,u_1\in X$ there is a unique solution $u\in C^1(]-T,T[;X)$.

It is well known that in the strictly hyperbolic case

$$a(t,x,\xi)\geqslant a_0|\xi|^2, \quad a_0>0, \ t\in]-T,T[,x,\xi\in\mathbf{R}^n \tag{1.4}$$

if the coefficients a_{ij} are Lipschitz continuous in the variable t, then the problem (1.1) is well-posed in the Sobolev spaces $H^{-\infty}(\mathbf{R}^n)=\bigcup_s H^s(\mathbf{R}^n)$ and $H^\infty(\mathbf{R}^n)=\bigcap_s H^s(\mathbf{R}^n)$. The C^∞ well-posedness follows by the existence of domains of dependence.

This may fail to be true either for a weakly hyperbolic equation, that is when $a(t,x,\xi)=0$ at some point $(t,x,\xi),\xi\neq 0$, even if $a_{ij}\in C^\infty$, or for a strictly hyperbolic equation with non-Lipschitz coefficients.

In the weakly hyperbolic case, the C^∞ well-posedness holds for an effectively hyperbolic operator and it is stable under any perturbation of the lower-order terms $b(t,x,D_x),c(t,x)$. Otherwise, the first-order term $b(t,x,\xi)$ has to satisfy Levi