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Elliptic relative equilibria in the N-body problem☆

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Abstract

A planar central configuration of the N-body problem gives rise to a solution where each particle moves on a specific Keplerian orbit while the totality of the particles move on a homothety motion. If the Keplerian orbit is elliptic then the solution is an equilibrium in pulsating coordinates so we call this solution an *elliptic relative equilibrium*.

The totality of such solutions forms a four-dimensional symplectic subspace and we give a symplectic coordinate system which is adapted to this subspace and its symplectic complement. In our coordinate system, the linear variational equations of such a solution decouple into three subsystems. One subsystem simply gives the motion of the center of mass, another is Kepler's problem and the third determines the nontrivial characteristic multipliers.

Using these coordinates we study the linear stability of the elliptic relative equilibrium defined by the equilateral triangular central configuration of the three-body problem. We reproduce the analytic studies of G. Roberts. We also study the linear stability of the four- and five-body problem where three or four bodies of unit mass are at the vertices of a equilateral triangle or square and the remaining body is at the center with arbitrary mass μ.

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1. Introduction

Let $q_1, \dots, q_N \in \mathbb{R}^2$ be the position vectors, $p_1, \dots, p_N \in \mathbb{R}^2$ the momentum vectors of N particles of masses $m_1, \dots, m_N$ in an inertial (sidereal) frame. Let the distance between the jth and kth particles be denoted by $d_{jk} = \| q_j - q_k \|$. In these coordinates the *Hamiltonian*, H, and the *self-potential*, S, for the N-body problem are

$$H = \sum_{j=1}^{N} \frac{\| p_j \|^2}{2m_j} - S(q_1, \dots, q_N), \quad S = \sum_{1 \leqslant j < k \leqslant N} \frac{m_j m_k}{d_{jk}} \tag{1}$$

and the equations of motion are

$$\dot{q}_j = p_j / m_j, \quad \dot{p}_j = \frac{\partial S}{\partial q_j}, \quad j = 1, \dots, N. \tag{2}$$

A *central configuration* is a solution $q_1 = a_1, \dots, q_N = a_N$ of the algebraic equations

$$-\lambda m_j q_j = \frac{\partial S}{\partial q_j} (q_1, \dots, q_N) \tag{3}$$

for some constant λ. One shows that $\lambda = S(a)/2I(a) > 0$ where $I = \frac{1}{2} \sum m_j \|a_j\|^2$ is the moment of inertia.

Only the planar N-body problem is considered here and so sometimes we will think of vectors in $\mathbb{R}^2$ as complex numbers, i.e. we will identify $\mathbb{R}^2$ and $\mathbb{C}$ in the usual way. A classical and elementary result [10,12,15,20] is

Proposition 1.1. *Let $a_1, \dots, a_N$, $a_i \in \mathbb{C}$ be a central configuration with constant λ. Let $(z(t), Z(t)) \in \mathbb{C}^2$ be a solution of the Kepler problem (central force problem) with Hamiltonian*

$$H_{\mathrm{K}} = \frac{1}{2} \|Z\|^2 - \lambda / \|z\|, \quad z, Z \in \mathbb{R}^2. \tag{4}$$

Then

$$q_i = z(t) a_i, \quad p_i = m_i Z(t) a_i, \quad i = 1, \dots, N$$

is a solution of the N-body problem.