Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Differential Equations 214 (2005) 65–91

Journal of Differential Equations

www.elsevier.com/locate/jde

Strong solutions for differential equations in abstract spaces

Eduardo V. Teixeira

Department of Mathematics, University of Texas at Austin, RLM 9.136, Austin, Texas 78712-1082, USA

Received 25 April 2004; revised 29 October 2004

Available online 28 December 2004

Abstract

Let $(E, \mathscr{F})$ be a locally convex space. We denote the bounded elements of E by $E_b := \{x \in E : \|x\|_{\mathscr{F}} = \sup_{\rho \in \mathscr{F}} \rho(x) < \infty\}$. In this paper, we prove that if B_{E_b} is relatively compact with respect to the $\mathscr{F}$ topology and $f : I \times E_b \to E_b$ is a measurable family of $\mathscr{F}$-continuous maps then for each $x_0 \in E_b$ there exists a norm-differentiable, (i.e. differentiable with respect to the $\| \cdot \|_{\mathscr{F}}$ norm) local solution to the initial valued problem $u_t(t) = f(t, u(t))$, $u(t_0) = x_0$. All of this machinery is developed to study the Lipschitz stability of a nonlinear differential equation involving the Hardy–Littlewood maximal operator.

MSC: primary 45N05; 45G10; 65L05; secondary 46A03; 46B10; 46B50

Keywords: Differential equations in locally convex spaces; Strong solutions; Regularity theory

1. Introduction

Differential equations modeled in Banach spaces have attracted the attention of many researchers throughout the last century. Most of their efforts were concentrated in the study of the classical Cauchy problem, also called the initial value

E-mail address: teixeira@math.utexas.edu.

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.11.006

problem (IVP)

$$\begin{cases} u_t(t) = f(t, u(t)) \text{ in } (a, b), \\ u(a) = u_0. \end{cases} \tag{1.1}$$

The map f is a 1-parameter family of fields between a Banach space, i.e., $f\colon [a, b] \times E \to E$. The theory of differential equations in Banach spaces has provided clever and useful strategies to study many problems that appear in both applied and abstract mathematics. The most common applications concern partial differential equations on the Euclidean spaces which arise from physical systems.

Let X be a Banach space and $F\colon [a, b] \times X \to X$ be continuous. It is well known that if either $\dim X < \infty$ or F is Lipschitz, then for each pair $(t_0, x_0) \in [a, b] \times X$, there exists a C^1-curve $x : (t_0 - \delta, t_0 + \delta) \to X$ such that $x(t_0) = x_0$ and $x_t(t) = F(t, x(t))$. Dieudonné in [11] provided the first example of a continuous map from an infinite dimensional Banach space for which there is no solution to the related IVP. In his simple and insightful example, $X = c_0$ and $F(x_1, x_2, \dots) := (|x_n|^{1/2} + 1/n)$. He noticed that there is no solution for the IVP $x(0) = 0$, $x_t(t) = F(x(t))$. Yorke [37] gave an example of the same phenomena in a Hilbert space. Afterwards, Godunov in [16] proved that for every infinite dimensional Banach space, there exists a continuous field such that there is no solution to the related initial valued problem. It turned out that continuity was not the right assumption on the field F. Many celebrated works have been developed since the 1970s in order to obtain suitable extensions for the continuity notion on finite dimensional spaces. Basically two branches were born on this journey: uniformly continuity and continuity in the weak topology. The former came from the observation that if $R_0 := [a, b] \times \overline{B}_X(x_0, r)$, $F\colon R_0 \to X$ is continuous and if $\dim X < \infty$, F is automatically uniformly continuous, due to the compactness of R_0. For reference in this type of research direction, i.e., strong topology assumptions, we cite, for instance, [22,25]. The latter came from one of the most fruitful ideas in functional analysis. Weak topology appeared as an attempt to grapple with the lack of local compactness in infinite dimensional Banach spaces. If the Banach space X is reflexive, we recover locally compactness by endowing it with the weak topology. We observe that the weak topology coincides with the strong topology in a Banach space X if, and only if, $\dim X < \infty$.

The first paper related to the existence of weak solutions for differential equations in Banach spaces relative to the weak topology was [33]. Its main result is

Theorem 1.1 (*Szep*). *Let E be a reflexive Banach space and f be a weak–weak continuous function on $P = \{t_0 \leqslant t \leqslant t_0 + a, \|x - x_0\| \leqslant b\}$. Let $\|f(t, x)\| \leqslant M$ on P. Then the IVP $x' = f(t, x)$, $x(t_0) = x_0$ has at least one weak solution defined on $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/M)$.*

Chow and Schuur in [6] treat the case where E is separable and reflexive and $f\colon (0, 1) \times E \to E$ is a weak continuous function with bounded range. The next step was given by Kató in [18]. In this paper he observed that if $f\colon [0, T] \times \overline{B}_E(u_0, r) \to E$