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Clustered spots in the FitzHugh–Nagumo system

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Abstract

We construct clustered spots for the following FitzHugh–Nagumo system:

$$\begin{cases} \varepsilon^2 \Delta u + f(u) - \delta v = 0 & \text{in } \Omega, \\ \Delta v + u = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth and bounded domain in R^2. More precisely, we show that for any given integer K, there exists an $\varepsilon_K > 0$ such that for $0 < \varepsilon < \varepsilon_K$, $\varepsilon^{m'} \leqslant \delta \leqslant \varepsilon^m$ for some positive numbers m', m, there exists a solution $(u_\varepsilon, v_\varepsilon)$ to the FitzHugh–Nagumo system with the property that u_ε has K spikes $Q_1^\varepsilon, \ldots, Q_K^\varepsilon$ and the following holds:

(i) The center of the cluster $\frac{1}{K}\sum_{i=1}^K Q_i^\varepsilon$ approaches a hotspot point $Q_0 \in \Omega$.

(ii) Set $l^\varepsilon = \min_{i \neq j} |Q_i^\varepsilon - Q_j^\varepsilon| = \frac{1}{\sqrt{a}} \log\left(\frac{1}{\delta\varepsilon^2}\right) \varepsilon(1 + o(1))$. Then $(\frac{1}{l^\varepsilon} Q_1^\varepsilon, \ldots, \frac{1}{l^\varepsilon} Q_K^\varepsilon)$ approaches an optimal configuration of the following problem:

(*) *Given K points $Q_1, \ldots, Q_K \in R^2$ with minimum distance* 1, *find out the optimal configuration that minimizes the functional* $\sum_{i \neq j} \log |Q_i - Q_j|$.

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1. Introduction

In this paper, we study the steady-states for the FitzHugh–Nagumo system [14,22]. This is a two-variable reaction–diffusion system derived from the Hodgkin–Huxley model for nerve-impulse propagation [18]. In a suitably rescaled fashion it can be written as follows:

$$\text{(FN)} \begin{cases} u_t = \varepsilon^2 \Delta u + f(u) - v & \text{in } \Omega, \\ v_t = \Delta v - \delta\gamma v + \delta u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The unknowns $u = u(x,t)$ and $v = v(x,t)$ represent the electric potential and the ion concentration across the cell membrane at a point $x \in \Omega \subset R^N$ $(N = 1, 2, \ldots)$ and at a time $t > 0$, respectively; $\varepsilon > 0$, $\delta > 0$, and $\gamma > 0$ are real constants; $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N; Ω is a smooth bounded domain in R^N; $f(u) = u(1-u)(u-a)$ with $a \in (0, \frac{1}{2})$.

In this paper, we consider steady-states of FN, namely we study the following elliptic system:

$$\begin{cases} \varepsilon^2 \Delta u + f(u) - \delta v = 0 & \text{in } \Omega, \\ \Delta v - \delta\gamma v + \delta u = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For simplicity, from now on we assume that $\gamma = 0$. (With slight modifications, the results also hold for fixed $\gamma > 0$.) Setting $v = \delta\tilde{v}$ and dropping the tilde we get the system

$$\begin{cases} \varepsilon^2 \Delta u + f(u) - \delta v = 0 & \text{in } \Omega, \\ \Delta v + u = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This is the final form of the system which we will study in the rest of the paper.

In the investigation of system (1.1) we make use of the fact that it arises as the Euler–Lagrange equation to the energy functional $E_\varepsilon : H_0^1(\Omega) \to R$ given by

$$E_\varepsilon[u] = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u) + \frac{\delta}{2} \int_\Omega u T[u], \tag{1.3}$$

where $F(u) = \int_0^u f(s)\, ds$. Here $v = T[u]$ for given $u \in L^2(\Omega)$ is defined as the unique solution $v \in H^2(\Omega)$ of the linear problem

$$\Delta v + u = 0 \text{ in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$