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The instability of spiky steady states for a competing species model with cross diffusion

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Abstract

This paper is concerned with the stability/instability of a class of positive spiky steady states for a quasi-linear cross-diffusion system describing two-species competition. By detailed spectral analysis, it is proved that the spiky steady states for the related shadow system are linearly unstable and the spiky steady states for the original cross-diffusion system are non-linearly unstable.

MSC: 35B25; 35B35; 35K20; 35K55; 35K57

Keywords: Cross-diffusion systems; Spiky steady states; Instability; Spectral analysis

0. Introduction

In order to investigate the spatial segregation under inter- and intra-population pressure, Shigesada et al. [13] proposed a two-species competition model with self- and cross-diffusion, which can be simply described by

$$\begin{cases} u_t = \triangle[(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), & x \in \Omega, \quad t > 0, \\ v_t = \triangle[(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2u - c_2v), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{0.1}$$

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where u and v are the population densities of two competing species, $a_j, b_j, c_j,\ d_j$ $(j = 1, 2)$ are all positive constants, the self-diffusion rates ρ_{jj} $(j = 1, 2)$, and cross-diffusion rates ρ_{ij} $(i, j = 1, 2)$ are non-negative. If $\rho_{ij} = 0$ $(i, j = 1, 2)$, (1) is reduced to the typical Lotka–Volterra competition model, which has been extensively and deeply studied in the past several decades.

For the initial boundary value problem of (1) with cross-diffusion terms and with general reaction terms, by the abstract theory established in [1], it is well known that system (1) is related with analytic semigroup and can still be considered as a parabolic system. The existence of global solutions in time, and the existence and stability of steady states or travelling waves has been investigated by many authors (see [4–11,15–17] and the references therein).

Especially for the Neumann boundary value problem (1), it is well known that for non-cross-diffusion cases ($\rho_{12} = \rho_{21} = 0$), and Ω being convex, there exists no stable non-constant steady state. While many theoretical and numerical results indicate that the appearance of cross-diffusion terms in (1) induce some new pattern formation, which seems interesting theoretically and biologically. In [10], for the case

$$n = 1, \quad \rho_{11} = \rho_{21} = \rho_{22} = 0, \quad \text{and} \quad \frac{1}{4}\frac{b_1}{b_2} + \frac{3}{4}\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2},$$

it was shown that when d_2 small enough, d_1 and ρ_{12}/d_1 large enough, there exist steady states with transition layers for (1), which was proved to be stable by the SLEP method [5].

In [6,7], the existence and non-existence of non-constant steady states of (1) with self- and cross-diffusion terms are widely investigated. Here we only mention that in the following two cases:

$$Case\ \text{I:} \quad \frac{1}{2}\left(\frac{b_1}{b_2} + \frac{c_1}{c_2}\right) < \frac{a_1}{a_2} < \frac{1}{4}\frac{b_1}{b_2} + \frac{3}{4}\frac{c_1}{c_2},$$

or

$$Case\ \text{II:} \quad \frac{1}{4}\frac{b_1}{b_2} + \frac{3}{4}\frac{c_1}{c_2} < \frac{a_1}{a_2} < \frac{1}{2}\left(\frac{b_1}{b_2} + \frac{c_1}{c_2}\right),$$

it was shown in [7] that as $d_2 > 0$ is small enough, and ρ_{12}/d_1 and ρ_{12} large enough, there exist positive steady states with spike layers. There are some other recent results on the existence of non-constant steady states [9].

As for the existence and stability of steady states with spike layers, there is a lot of recent research work on some reaction–diffusion systems related with Turing's pattern, such as Gierer–Meinhardt systems, Gray–Scott systems, etc. (see [2,3,12,14] and references therein). There are mainly two kinds of methods in the recent study of stability of spiky steady states, one is closely related with shadow system analysis [12], the other is the NLEP method (for short) [3,14].