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C^1-stably expansive flows

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Abstract

In this paper, the C^1 interior of the set of vector fields whose integrated flows are expansive is characterized as the set of vector fields without singularities satisfying both Axiom A and the quasi-transversality condition, and it is proved that the above vector fields possessing the shadowing property must be structurally stable. As a corollary, there exists a non-empty C^1 open set of vector fields whose integrated flows do not have the shadowing property.

MSC: 37C27; 37C50; 37D20

Keywords: Expansive flows; Vector fields; Shadowing property; Axiom A; Transversality condition; Structurally stable; Quasi-Anosov

1. Introduction

We are interested in characterizing the geometrical structure of dynamical systems possessing a topological property of Anosov systems such as topological stability under

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the C^1 open condition (see [9]). The C^1 open condition signifies that the topological property under consideration is preserved with respect to C^1 small perturbations of the system.

In this paper, we consider the set of expansive flows (vector fields), and investigate its geometric structure from the above point of view. More precisely, the C^1 interior of the set of vector fields whose integrated flows are expansive is characterized as the set of vector fields without singularities satisfying both Axiom A and the quasi-transversality condition. Furthermore, we prove that such vector fields possessing the shadowing property must be structurally stable. As a corollary, it follows from Robinson's example (see [14]) that there exists a non-empty C^1 open set of vector fields whose integrated flows do not have the shadowing property.

Let M be a C^∞ closed manifold, and denote by $\mathcal{X}^1(M)$ the set of C^1 vector fields on M endowed with the C^1 topology. Denote by $\mathcal{E}(M)$ the set of $X \in \mathcal{X}^1(M)$ whose integrated flow is expansive, and by $\operatorname{int}\mathcal{E}(M)$ the C^1 interior of $\mathcal{E}(M)$ in $\mathcal{X}^1(M)$.

The following result is obtained.

Theorem A. *For $X \in \mathcal{X}^1(M)$, the following conditions are mutually equivalent:*

(i) $X \in \operatorname{int}\mathcal{E}(M)$,
(ii) *X is quasi-Anosov,*
(iii) *X has no singularities, and satisfies both Axiom A and the quasi-transversality condition.*

A similar result is obtained by Mañé in [7,8] for diffeomorphisms on M. When $\dim M = 3$, it is easy to see that every quasi-Anosov vector field on M is Anosov. Thus, every $X \in \operatorname{int}\mathcal{E}(M)$ is Anosov when $\dim M = 3$. However, in higher dimensions that is not true by Robinson's example (see [14]).

In the present paper, we also prove the following.

Theorem B. *For $X \in \mathcal{X}^1(M)$, the following conditions are mutually equivalent:*

(i) *$X \in \operatorname{int}\mathcal{E}(M)$ and has the shadowing property,*
(ii) *$X \in \operatorname{int}\mathcal{E}(M)$ and is structurally stable,*
(iii) *X is Anosov.*

In [15] the second author showed an analogue of the above theorem for diffeomorphisms by making use of a result proved in [8].

Let $X \in \mathcal{X}^1(M^{11})$ be Robinson's example of a quasi-Anosov vector field that is not Anosov on an 11-dimensional manifold M^{11} (for diffeomorphisms, see [2]). Since the set of quasi-Anosov vector fields is C^1 open in $\mathcal{X}^1(M)$ (see Remark 1), it is easy to see that every C^1 nearby system $Y \in \mathcal{X}^1(M^{11})$ of X is also quasi-Anosov but not Anosov by construction. Thus, combining these facts with Theorem B we have the following.

Corollary. *There exists a non-empty C^1 open set $\mathcal{U} \subset \mathcal{X}^1(M^{11})$ whose any element does not have the shadowing property.*

Thus the set of vector fields having the shadowing property on M is not C^1 dense in $\mathcal{X}^1(M)$ in general.