Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Differential Equations 219 (2005) 40–77
Journal of Differential Equations
www.elsevier.com/locate/jde

Self-similar solutions of a semilinear parabolic equation with inverse-square potential☆

Guillermo Reyes[a,*], Alberto Tesei[b]

[a]*Escuela Politécnica Superior, Universidad Carlos III de Madrid, Ave. de la Universidad, 30, 28911 Madrid, Spain*
[b]*Dipartimento di Matematica "G. Castelnuovo". Università di Roma, "La Sapienza", P.le A. Moro 5, I-00185 Roma, Italy*

Received 2 August 2004; revised 23 June 2005

Available online 19 September 2005

Abstract

We investigate existence, nonexistence and asymptotical behaviour—both at the origin and at infinity—of radial self-similar solutions to a semilinear parabolic equation with inverse-square potential. These solutions are relevant to prove nonuniqueness of the Cauchy problem for the parabolic equation in certain Lebesgue spaces, generalizing the result proved by Haraux and Weissler [Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J. 31 (1982) 167–189] for the case of vanishing potential.

1. Introduction

In this paper we investigate existence, nonexistence and asymptotical behaviour of nonnegative solutions to the ordinary differential equation

$$(Pf')' + \left(\frac{c}{\xi^2} + \frac{1}{q-2}\right) Pf + P|f|^{q-2} f = 0 \tag{1.1}$$

☆ Work partially supported through RTN Contract HPRN-CT-2002-00274.
* Corresponding author.
E-mail addresses: guillermo.reyes@uc3m.es, greyes@math.uc3m.es (G. Reyes), tesei@mat.uniroma1.it (A. Tesei).

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2005.06.031

in $\mathbb{R}_+$, where

$$P(\xi) := \xi^{n-1} e^{\frac{\xi^2}{4}}, \tag{1.2}$$

$n \geqslant 3, q > 2$ and the coefficient c satisfies the inequality $0 < c < c_0$, $c_0 := \frac{(n-2)^2}{4}$ denoting the best constant in the Hardy inequality.

(a) Eq. (1.1) arises in the analysis of radial self-similar solutions to the semilinear parabolic equation with inverse-square potential

$$v_t = \Delta v + \frac{c}{r^2} v + |v|^{q-2} v \tag{1.3}$$

in $S := \mathbb{R}^n \times \mathbb{R}_+$ ($n \geqslant 3$), where $r \equiv |x|$ and c, q are as above. Such solutions are of the form

$$v(x,t) = t^{-\frac{1}{q-2}} f(r/\sqrt{t}).$$

Upon substitution into (1.3), it is easily seen that the profile f satisfies Eq. (1.1) in $\mathbb{R}_+$, where $\xi := r/\sqrt{t}$ and $' \equiv d/d\xi$.

In the following an important role is played by the roots:

$$\lambda = \lambda_\pm := 2 - n \pm 2\sqrt{c_0 - c} \tag{1.4}$$

of the equation

$$\lambda^2 + 2(n-2)\lambda + 4c = 0 \tag{1.5}$$

(e.g., see Theorem 1.6 below; observe that $\lambda_- < 2-n < \lambda_+ < 0$). These roots naturally appear when we perform in Eq. (1.1) the change of unknown $f(\xi) = \xi^{\lambda/2} g(\xi)$; in fact, the choice $\lambda = \lambda_\pm$ gives the following equation for g:

$$(Hg')' - \sigma H g + K|g|^{q-2} g = 0. \tag{1.6}$$

Here

$$H(\xi) := \xi^{\lambda+n-1} e^{\frac{\xi^2}{4}} = \xi^\lambda P(\xi), \tag{1.7}$$

$$K(\xi) := \xi^{\frac{\lambda q}{2}+n-1} e^{\frac{\xi^2}{4}} = \xi^{\frac{\lambda}{2}(q-2)} H(\xi), \tag{1.8}$$