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Convergence in competition models with small diffusion coefficients

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Abstract

It is well known that for reaction–diffusion 2-species Lotka–Volterra competition models with spatially independent reaction terms, global stability of an equilibrium for the reaction system implies global stability for the reaction–diffusion system. This is not in general true for spatially inhomogeneous models. We show here that for an important range of such models, for small enough diffusion coefficients, global convergence to an equilibrium holds for the reaction–diffusion system, if for each point in space the reaction system has a globally attracting hyperbolic equilibrium. This work is planned as an initial step towards understanding the connection between the asymptotics of reaction–diffusion systems with small diffusion coefficients and that of the corresponding reaction systems.

Keywords: Reaction–diffusion; Competing species; Spatial inhomogeneity; Small diffusion limit; Asymptotic dynamics

1. Introduction

Given an arbitrary system of reaction–diffusion equations, in general the asymptotic behavior of the corresponding reaction system gives little information

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on the asymptotic behavior of the reaction–diffusion systems. Suppose, however, that the reaction system has a globally asymptotically stable equilibrium, in the sense that this property holds for each point x of the spatial domain. Does this provide more information? In particular does it imply that the corresponding reaction–diffusion system has the same property? That it is too optimistic to expect this is clear from recent intriguing work [29,36]. In [29] for example, it is shown that there is a class of system consisting of a pair of simultaneous reaction–diffusion equations with homogeneous reaction term (i.e. independent of x) with the following property. Given any unequal diffusion coefficients μ, ν, there is a choice of initial conditions such that the solution blows up in finite time. In view of these results, in what direction should one look?

It is well known [9] that for large μ, ν if the reaction–diffusion system has a L^∞ bounded positively invariant set, for initial conditions in that set, asymptotically the orbits are close to those of the reaction system for the spatially averaged solution. For further investigation in this area one may refer to [8,15,16]. It would be extremely useful if one could extend the class of reaction–diffusion equations for which the reaction system provides useful information. The obvious direction to look is to small μ, ν, where we might hope for global convergence of the reaction–diffusion system to an equilibrium 'close' to that of the corresponding reaction system, but in view of the above-mentioned results, we must further restrict the class of equations.

It is difficult to predict the general direction in which we should look, but we may note that at least for some simple situations, the result is true, in fact for arbitrary μ, ν. One such situation is the following Lotka–Volterra system with α, β, b and c positive (by which is always meant strictly positive) constants and zero Neumann boundary conditions:

$$\begin{cases} u_t = \mu \Delta u + u[\alpha - u - bv], \\ v_t = \nu \Delta v + v[\beta - cu - v]. \end{cases} \tag{1.1}$$

Of course this is also true for the analogous predator–prey system. This suggests that it is reasonable to enquire whether a general competing species model would have this property; in view of the considerable recent interest in spatially inhomogeneous models [2–7,10–13,17,19,21–24,27,30,32] a result along these lines would be of importance. Similarly, the property might also be true for general predator–prey and other models, such as the migration–selection model from population genetics [28], but it is likely to be much harder to prove since a monotonicity structure cannot be invoked, see Section 5 for further discussion.

Consider then the following general reaction–diffusion system

$$\begin{cases} u_t = \mu \Delta u + uf(u, v, x), & \\ v_t = \nu \Delta v + vg(u, v, x) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \;\; v(x, 0) = v_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{1.2}$$