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Existence and blow up of small-amplitude nonlinear waves with a sign-changing potential

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Abstract

We study the nonlinear wave equation with a sign-changing potential in any space dimension. If the potential is small and rapidly decaying, then the existence of small-amplitude solutions is driven by the nonlinear term. If the potential induces growth in the linearized problem, however, solutions that start out small may blow-up in finite time.

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1. Introduction

Consider the nonlinear wave equation with potential

$$\begin{cases} \partial_t^2 u - \Delta u + V(x) \cdot u = F(u) & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = \varphi(x); \quad \partial_t u(x,0) = \psi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $V(x)$ is some known function and $F(u)$ behaves like $|u|^p$ for some $p > 1$. When it comes to the special case $V(x) \equiv 0$, this equation has been extensively

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studied since Fritz John's seminal work [8]. For that case, in particular, the existence of small-amplitude solutions is known to depend on both the exact value of p and the decay rate of the initial data. In this paper, we address the more general case (1.1) when the potential $V(x)$ is of arbitrary sign. Our aim is to show that the existence of small-amplitude solutions may also be affected by two additional parameters, namely, the amplitude and the decay rate of the potential $V(x)$.

First, consider solutions to (1.1) when $V(x) \equiv 0$ and the small initial data have compact support. John's classical result [8] in $n = 3$ space dimensions ensures their global existence if $p > 1+\sqrt{2}$ and their blow-up if $1 < p < 1+\sqrt{2}$. More generally, a similar dichotomy holds in $n \geqslant 2$ space dimensions, where the borderline case is given by the positive root p_n of the quadratic

$$(n-1)p_n^2 = (n+1)p_n + 2; \tag{1.2}$$

see [4,6–8,17,22,28,33]. As for the borderline case $p = p_n$ with $n \geqslant 2$, the blow-up of solutions persists [21,32]. Finally, when $n = 1$, blow-up occurs for any $p > 1$; see [10].

Next, consider solutions to (1.1) when $V(x) \equiv 0$ and the small initial data decay slowly. In $n = 2, 3$ space dimensions, their global existence is ensured as long as $p > p_n$ and the initial data satisfy

$$\sum_{|\alpha| \leqslant 3} |\partial_x^\alpha \varphi(x)| + \sum_{|\alpha| \leqslant 2} |\partial_x^\alpha \psi(x)| \leqslant \varepsilon (1+|x|)^{-k-1} \tag{1.3}$$

for some $k \geqslant 2/(p-1)$ and some small $\varepsilon > 0$. On the other hand, blow-up may occur for any $p > 1$ when the initial data are such that

$$\varphi(x) = 0, \quad \psi(x) \geqslant \varepsilon (1+|x|)^{-k-1} \qquad \text{in } \mathbb{R}^n \tag{1.4}$$

for some $0 \leqslant k < 2/(p-1)$ and $\varepsilon > 0$; see [1,2,15,29–31]. In $n \geqslant 4$ space dimensions, the same blow-up result holds, provided that φ, ψ are radially symmetric [26,27]. However, the existence result is slightly modified as follows. Instead of (1.3), one assumes that

$$\sum_{|\alpha| \leqslant 2} \langle x \rangle^{|\alpha|} \, |\partial_x^\alpha \varphi(x)| + \sum_{|\alpha| \leqslant 1} \langle x \rangle^{|\alpha|+1} \, |\partial_x^\alpha \psi(x)| \leqslant \varepsilon \, \langle x \rangle^{-k}, \tag{1.5}$$

where φ, ψ are radially symmetric and $\langle x \rangle = 1+|x|$ for each $x \in \mathbb{R}^n$. When $k \geqslant 2/(p-1)$ and $\varepsilon > 0$ is small, one then has global solutions in $n \geqslant 4$ space dimensions as well [12,14], although the case $2/(p-1) \neq k > n/2$ has only been treated as part of the even-dimensional scattering results of [14].

In the remaining of this paper, we shall mostly focus on the radially symmetric version of the nonlinear wave equation with potential (1.1). Thus, the equation of