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Asymptotic smoothing and attractors for the generalized Benjamin–Bona–Mahony equation on R^3

Milena Stanislavova[a,1], Atanas Stefanov[a,2], Bixiang Wang[b,*,3]

[a]Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA
[b]Department of Mathematics, New Mexico Institute of Mining and Technology, Socorro, NM 87801, USA

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Abstract

We study the asymptotic behavior of the solutions of the Benjamin–Bona–Mahony equation defined on $\mathbf{R}^3$. We first provide a sufficient condition to verify the asymptotic compactness of an evolution equation defined in an unbounded domain, which involves the Littlewood–Paley projection operators. We then prove the existence of an attractor for the Benjamin–Bona–Mahony equation in the phase space $H^1(\mathbf{R}^3)$ by showing the solutions are point dissipative and asymptotic compact. Finally, we establish the regularity of the attractor and show that the attractor is bounded in $H^2(\mathbf{R}^3)$.

MSC: Primary 35B40; secondary 35B41

Keywords: Benjamin–Bona–Mahony equation; Global attractor; Asymptotic behavior

* Corresponding author. Fax: +1 505 835 5366.
E-mail address: bwang@nmt.edu (B. Wang).

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1. Introduction and main results

In this paper, we are interested in the asymptotic behavior of the solutions of the following Benjamin–Bona–Mahony (BBM) equation defined on $\mathbf{R}^3$:

$$u_t - \Delta u_t - v\Delta u + div(f(u)) = g, \tag{1}$$

where v is a positive constant, $g \in L^2(\mathbf{R}^3)$ and $f(u) = u + \frac{1}{2}u^2$.

The BBM equation was proposed in [12] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions for this equation were studied by many authors, see, for example, [7,8,12,14,15,17,22,35,36,43]. The decay rates of solutions were investigated in [3–5,13,34,50] and the references therein. When the equation is defined in a bounded domain, the existence and finite dimensionality of the global attractor were proved in [6,16,46,49]. The regularity of the global attractor was established in [47] when the forcing term $g \in H^k$ with $k \geqslant 0$, and the Gevrey regularity was proved in [18] when g is in a Gevrey class. The authors of [18] also proved the existence of two determining nodes for the one-dimensional equation with periodic boundary conditions.

The first goal of this paper is to establish the existence of attractors for the BBM equation on $\mathbf{R}^3$. In other words, we will prove the following result.

Theorem 1. *Given a ball $B = \{u \in H^1(\mathbf{R}^3) : \|u\|_{H^1} \leqslant R\}$. If g is small enough, then the BBM equation has an attractor $\mathcal{A}$ in $H^1(\mathbf{R}^3)$ which is a compact invariant set and attracts every bounded subset of B with respect to the norm topology of $H^1(\mathbf{R}^3)$.*

Remark. As stated in Theorem 1, the existence of the attractor for the BBM equation requires that the forcing term g satisfy a smallness condition, which depends on the size of the ball B. More precisely, for every $R > 0$, there exists $\varepsilon(R) > 0$, such that g satisfies the condition:

$$\int g\,dx = 0 \quad \text{and} \quad \|g\|_{L^2} + \int |x||g(x)|\,dx \leqslant \varepsilon(R),$$

where $\varepsilon(R) \to 0$ as $R \to \infty$. For more details about the smallness condition on g, we refer the reader to Theorem 3.

Note that the domain of Eq. (1) is unbounded, which causes additional difficulty when we prove the existence of attractors because, in this case, the Sobolev embeddings are not compact. There are several methods which can be used to show the existence of attractors in standard Sobolev spaces when the equations are defined in unbounded domains. One method is to show that the weak asymptotic compactness is equivalent to the strong asymptotic compactness by an energy equation technique [28,31–33,38,42]. A second method is to decompose the solution operator into a compact part and an asymptotically small part [19–21]. A third method is to prove that the solutions are