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Exponential decay for the fragmentation or cell-division equation

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Abstract

We consider a classical integro-differential equation that arises in various applications as a model for cell-division or fragmentation. In biology, it describes the evolution of the density of cells that grow and divide. We prove the existence of a stable steady distribution (first positive eigenvector) under general assumptions in the variable coefficients case. We also prove the exponential convergence, for large times, of solutions toward such a steady state.

Keywords: Cell division; Fragmentation; Asymptotic stability

1. Introduction

This paper is concerned with the equation

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + b(x)\, n(t,x) = 4b(2x)\, n(t,2x), & t > 0,\ x \geqslant 0, \\ n(t, x=0) = 0, & t > 0, \\ n(0,x) = n^0(x) \in L^1(\mathbb{R}^+). \end{cases} \tag{1}$$

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It arises in many applications, in particular as a basic model for size-structured populations: $n(t,x)$ denotes the population density of cells of size x at time t. The cells grow at a constant rate but also divide into two cells of equal size at a rate $b(x)$ when mitosis occurs (see [13] for a classical reference on the subject and [3] for a more recent application to cell division, [15] for age/maturation models and further references). This model also appears in physics to describe a fragmentation (degradation) phenomenon in polymers, droplets ([12,9,1] and references therein) and in telecommunications systems to describe some internet protocols [2]. Several variants of the model are possible and we refer to [4] for a probabilistic study of a model with the same feature (competition between a 'growth process' and a 'fragmentation process').

Our first purpose is to study existence of the first eigenvector $N(x)$

$$\begin{cases} \frac{\partial}{\partial x} N(x) + (\lambda + b(x))\ N(x) = 4b(2x)\ N(2x), \quad x \geqslant 0, \\ N(0) = 0, \qquad N(x) > 0 \quad \text{for } x > 0, \qquad \int_0^\infty N(x)\,dx = 1, \end{cases} \tag{2}$$

with λ the first eigenvalue (sometimes called the Malthus parameter in biology). Then, and our second purpose is to make this rigorous, one can expect that this density plays the role of a so-called stable steady distribution, that is, after a time renormalization, all the solutions to (1) converge to a multiple of N. To be more precise, we need to introduce the dual operator

$$\begin{cases} \frac{\partial}{\partial x}\psi(x) - (\lambda + b(x))\ \psi(x) = -2b(x)\ \psi(\frac{x}{2}), \quad x \geqslant 0, \\ \psi(x) > 0 \quad \text{for } x \geqslant 0, \qquad \int_0^\infty N(x)\psi(x)\,dx = 1. \end{cases} \tag{3}$$

It plays a fundamental role in the dynamics of (1) because its solution allows to define a conservation law for the evolution equation for $n(t,x)$:

$$\int_0^\infty n(t,x)e^{-\lambda t}\ \psi(x)\,dx = \int_0^\infty n^0(x)\psi(x)\,dx := \langle n^0 \rangle. \tag{4}$$

The construction of the eigenvalue λ and eigenfunctions N, ψ as well as the long-time asymptotics face some technical difficulties although the former is a variant of the Krein–Rutman theorem (see [7] for instance for a recent presentation and different versions). First, we work on the half line that lacks compactness, second, the regularizing effect (positivity) of the division term is very indirect and the fact that all the derivatives $N^{(p)}(x=0)$ vanish is a specific difficulty. Therefore, we base our results on (i) original a priori estimates for the steady states N, ψ and (ii) a perturbation method around the constant coefficient case which is simpler for the exponential convergence. Indeed, we have the following.