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# Local existence with physical vacuum boundary condition to Euler equations with damping

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**Abstract**

In this paper, we consider the local existence of solutions to Euler equations with linear damping under the assumption of physical vacuum boundary condition. By using the transformation introduced in Lin and Yang (Methods Appl. Anal. 7 (3) (2000) 495) to capture the singularity of the boundary, we prove a local existence theorem on a perturbation of a planar wave solution by using Littlewood–Paley theory and justifies the transformation introduced in Liu and Yang (2000) in a rigorous setting.


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*Keywords:* Euler equations; Physical vacuum boundary condition; Littlewood–Paley theory; Local existence

## 1. Introduction

In this paper, we are interested in the time evolution of a gas connecting to vacuum with physical boundary condition. By assuming that the governed equations for the gas dynamics are Euler equations with linear damping, cf. [16] for physical interpretation, one can see that the system fails to be strictly hyperbolic at the

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vacuum boundary because the characteristics of different families coincide. As discussed in the previous works, cf. [5,11–13], the canonical vacuum boundary behavior is the case when the space derivative of the enthalpy is bounded but not zero. In this case, the pressure has its non-zero finite effect on the evolution of the vacuum boundary. However, for this canonical (physical) case, the system becomes singular in the sense that it cannot be symmetrizable with regular coefficients so that the local existence theory for the classical hyperbolic systems cannot be applied. Furthermore, the linearized equation at the boundary gives a Keyldish-type equation for which general local existence theory is still not known. Notice that this linearized equation is quite different from the one considered in [18] for weakly hyperbolic equation which is of Tricomi type. To capture this singularity in the nonlinear settting, a transformation was introduced in [13] and some local existence results for bounded domain were also discussed. The transformed equation is a second-order nonlinear wave equation of an unknown function $\phi(y,t)$ with coefficients as functions of $y^{-1}\phi(y,t)$ and $\phi(0,t)\equiv 0$. Along the vacuum boundary, the physical boundary condition implies that the coefficients are functions of $\phi_y(0,t)$ which are bounded and away from zero. Hence, the wave equation has no singularity or degeneracy. However, its coefficients have the above special form so that the local existence theory developed for the classical nonlinear wave equation cannot be applied directly [8,9]. There are other works on this system with vacuum, please refer to [6,10] etc. and references therein.

Even though a transformation to capture the singularity in the physical boundary condition at vacuum interface is introduced in [13], the energy method presented there may not give a rigorous proof of the existence theory, especially in the general setting. It is because the coefficients in the reduced wave equation which are power functions of $y^{-1}\phi$ correspond to the fractional differentiations of $\phi$. Under this consideration, we think the application of Littlewood–Paley theory based on Fourier theory is more appropriate. Therefore, as the first step in this direction, in this paper we will study the local existence of solutions satisfying the physical boundary condition when the initial data is a small perturbation of a planar wave solution where the enthalpy is linear in the space variable, [11]. By applying the Littlewood–Paley theory, we obtain the solution local in time with the prescribed physical boundary condition.

Precisely, we consider the one-dimensional compressible Euler equations for isentropic flow with damping in Eulerian coordinates

$$\begin{aligned}&\rho_t+(\rho u)_x=0,\\&\rho u_t+\rho u u_x+p(\rho)_x=-\rho u,\end{aligned}\tag{1.1}$$

where $\rho$, $u$ and $p(\rho)$ are density, velocity and pressure, respectively. And the linear frictional coefficient is normalized to 1. When the initial density function contains vacuum, the vacuum boundary $\Gamma$ is defined as

$$\Gamma=\mathrm{cl}\{(\vec{x},t)\mid\rho(\vec{x},t)>0\}\cap\mathrm{cl}\{(\vec{x},t)\mid\rho(\vec{x},t)=0\}.$$