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Pseudo-normal form near saddle-center or saddle-focus equilibria

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Abstract

In this paper we introduce the pseudo-normal form, which generalizes the notion of normal form around an equilibrium. Its convergence is proved for a general analytic system in a neighborhood of a saddle-center or a saddle-focus equilibrium point. If the system is Hamiltonian or reversible, this pseudo-normal form coincides with the Birkhoff normal form, so we present a new proof in these celebrated cases. From the convergence of the pseudo-normal form for a general analytic system several dynamical consequences are derived, like the existence of local invariant objects.

MSC: 34C20; 34C14

Keywords: Convergence of normal forms; Hamiltonian systems; Reversible systems

1. Introduction and main results

Since normal forms were introduced by Poincaré they have become a very useful tool to study the local qualitative behavior of dynamical systems around equilibria, see for instance [1,3,8] and references therein. In a few words, given a system

$$\dot{X} = F(X) = \Lambda X + \widehat{F}(X), \tag{1}$$

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around an equilibrium $X = 0$, where $\widehat{F}(X)$ denotes terms of order at least 2 in X, a general normal form procedure consists on looking for a (formal power series close to the identity) transformation $X = \Phi(\chi) = \chi + \widehat{\Phi}(\chi)$ in such a way that the new system $\dot{\chi} = \Phi^* F(\chi) =: N(\chi) = \Lambda\chi + \widehat{N}(\chi)$ becomes in *normal form*, that is, when $\widehat{N}$ contains only the so-named *resonant terms*, monomials whose powers are intimately related to the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ formed by the eigenvalues of the matrix Λ of system (1).

In this work, we will focus our attention on analytic vector fields and will be specially concerned with the convergence of the *normalizing transformation* Φ. There are two well-known cases where the convergence of the normalizing transformation follows just from the properties of the vector of characteristic exponents λ (see, for instance, [1, Chapter 5, Section 24]):

(i) when λ belongs to the Poincaré domain, that is, the convex hull of the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ in the complex plane does not contain the origin;
(ii) when λ belongs to the complementary of this domain, the so-called Siegel's domain, and satisfies a Diophantine condition.

In the first case, the Theorem of Poincaré–Dulac ensures the convergence of a normalizing transformation conjugating the original system to a system having only a finite number of resonant terms. In the second case, the Diophantine condition permits to bound the small divisors appearing in the normalizing transformation and its convergence is also derived (Siegel's Theorem). The original system is conjugated to its linear part.

Notice that in both cases of convergence the normal form is a *polynomial* or, in other words, the number of resonant terms is finite. However, non polynomial normal forms do arise in some important families of dynamical systems, like the Hamiltonian or the reversible ones, where the characteristic exponents always belong to the Siegel's domain since they come in pairs $\{\pm\lambda\}$. In these cases, convergence results depend not only on the location of the characteristic exponents and their arithmetical properties but also on the kind of formal normal form they exhibit.

In 1971, Bruno (see [2, Chapter II, Sections 3, 4]) provided sufficient and, in some particular sense, necessary conditions ensuring this convergence. He denominated them *condition* ω and *condition* A. The condition ω depends on arithmetic properties of the vector of characteristic exponents λ, and can be checked explicitly. On the contrary, condition A imposes a strong restriction on the normal form forcing it (up to all order!) to depend only on one or two scalar functions.

We refer the reader to Bruno's paper [2, pp. 173–175] for a detailed account of these conditions. For the purpose of this paper, it is enough to notice that there are very few cases where the fulfillment of condition A follows from the nature of the original system. The most famous case is provided by the framework of the Hamiltonian systems, where the normal form is called the *Birkhoff normal form* (BNF in short). Among them, condition ω is trivially satisfied when there are no small divisors between the main characteristic exponents, but this only happens for Hamiltonian systems with one or two degrees of freedom.