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On the number of zeros of Abelian integrals for a polynomial Hamiltonian irregular at infinity☆

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Abstract

Up to now, most of the results on the tangential Hilbert 16th problem have been concerned with the Hamiltonian regular at infinity, i.e., its principal homogeneous part is a product of the pairwise different linear forms. In this paper, we study a polynomial Hamiltonian which is not regular at infinity. It is shown that the space of Abelian integral for this Hamiltonian is finitely generated as a $\mathbb{R}[h]$ module by several basic integrals which satisfy the Picard–Fuchs system of linear differential equations. Applying the bound meandering principle, an upper bound for the number of complex isolated zeros of Abelian integrals is obtained on a positive distance from critical locus. This result is a partial solution of tangential Hilbert 16th problem for this Hamiltonian. As a consequence, we get an upper bound of the number of limit cycles produced by the period annulus of the non-Hamiltonian integrable quadratic systems whose almost all orbits are algebraic curves of degree $k+n$, under polynomial perturbation of arbitrary degree.

MSC: 34C07; 34C08; 37G15; 34M50

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1. Introduction

In this paper, we study the number of zeros of Abelian integral for a polynomial Hamiltonian which is irregular at infinity.

1.1. The tangential Hilbert 16th problem

Let $H(x, y)$, $f(x, y)$, $g(x, y)$ be polynomials in two-real variables and Γ_h the closed connected component of level set $\{(x, y)|H(x, y) = h\}$. Suppose that

$$\omega = f(x, y)\,dx + g(x, y)\,dy \tag{1.1}$$

is a real polynomial 1-form with degree $d = \max\{\deg f(x, y), \deg g(x, y)\}$. The Abelian integral is defined by

$$I(h) = I(h, H, \omega) = \oint_{\Gamma_h} \omega. \tag{1.2}$$

The tangential Hilbert 16th problem, or the weakened Hilbert 16th problem, posed by Arnold [A1,A2], is to place an upper bound $Z(\deg H, d)$ of the number of zeros of $I(h)$ on the maximal connected interval of existence of Γ_h, in terms of $\deg H$ and d.

The general result of solving the tangential Hilbert 16th problem was achieved by Varchenko [V] and Khovanskii [K], who proved independently the existence of $Z(\deg H, d)$, but no explicit expression of $Z(\deg H, d)$ has been obtained. Many authors have contributed to estimate or to give an explicit upper bound of the number of zeros of $I(h)$ for the cubic and quartic elliptic Hamiltonians $H = y^2 + p(x)$, see for instance Petrov [P1,P2,P3], Rousseau and Zoladek [RZ], Zhao and Zhang [ZZz], Liu [Lc] etc. In the paper [HI2], Horozov and Iliev gave a linear upper bound $Z(3, d) \leqslant 15d + 15$ for general cubic Hamiltonians. The authors of the paper [NY3] constructed a linear differential equation satisfied by $I(h)$ and obtained using the tools from [IY] an asymptotical exponential bound for the number of zeros of $I(h)$. More results of this problem will be recalled in Sections 1.2–1.4.

1.2. Abelian integrals and limit cycles

We briefly recall the connection between the tangential Hilbert 16th problem and the number of limit cycles of planar vector fields.

1.2.1. The polynomial perturbations of Hamiltonian systems

Consider the perturbed system

$$dH(x, y) + \varepsilon\omega = 0, \tag*{$(1.3)_\varepsilon$}$$